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ANALYSIS OF MODES IN AN UNSTABLE STRIP LASER RESONATOR.(U)

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THESIS

AFIT/GEP/PH/80D-7

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ANALYSIS OF MODES
IN AN UNSTABLE
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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

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2Lt USAF
Graduate Engineering Physics
December 1980

Approved for public release; distribution unlimited

Preface

I would like to give special thanks to Captain Mark Rogers for the use of his plotting routine and for supplying me with the Moore and McCarthy Code. I would also like to thank Dr. D. Lee of the AFIT Math department for supplying the error function subroutine.

Most of all, I would like to thank my advisor, Major John Erkkila for his many helpful suggestions, for spotting the really ellusive bugs, and for offering freely of his invaluable insight.

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List of Symbols

a_n	series function weighting constants
b_n	series function weighting constants
c_n	series function weighting constants
E( )	Fresnel integral
F	Ordinary Fresnel #
F_n	Series function
g_i	i^{th} mirror g parameter
g( )	Eigenfunction
G_n	Series function
h	relative amplitude of basis wave
H_n	Sum of functions F_n and G_n
i	$\sqrt{-1}$
I_{sat}	Saturation intensity
k	Wave #
L	Cavity length
m	Magnification
M_1	Mirror 1
M_2	Mirror 2
N	Number of terms in series of H_n's
N_f,NEQ	Effective Fresnel number
P	Power
R_1	Radius Curvature, M_1
R_2	Radius Curvature, M_2

u( )	Field distribution
x	displacement from optic axis
y_0	Stationary phase point
λ	Normalized eigenvalue
γ	Eigenvalue
ξ	Gain-factor

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Abstract

The mode eigenvalue equation for an unstable strip laser resonator is developed from scalar diffraction theory. The field distributions are expressed as a series and the integral is then evaluated using a first order approximation to the method of stationary phase. The resulting approximate closed form is rearranged to form an eigenvalue polynomial, the roots of which are the mode eigenvalues. Eigenfunction expressions are then developed using a second order approximation to the method of stationary phase. Modifications to these expressions are then made to account for the presence of uniform gain in the resonator.

The results of a computer program using the derived expressions are presented. Comparisons to previously published results are made for the bare cavity case, and results for the loaded cavity case follow.

ANALYSIS OF MODES
IN AN UNSTABLE
STRIP LASER RESONATOR

I. Introduction

Background

An unstable laser resonator is a resonator in which the geometric path of a paraxial ray traveling back and forth between the two mirrors is unbounded in an infinite number of passes. This is opposed to a stable resonator, in which the ray path is bounded. Any ray inside an unstable resonator will eventually take on a direction from which it will not come into contact with either mirror, and thus leave the cavity. In this type of resonator, the product of the resonator mirror g parameters, where

$$g_i \equiv 1 - \frac{L}{R_i} \qquad i=1,2$$

lies outside the stable range of

$$0 \leq g_1 g_2 \leq 1$$

The utility or benefits of unstable resonators, for instance large mode volume and minimally transmitting optics (Ref 10:353), require that some method of mode analysis be

available. Several methods are available, but have various drawbacks, such as excessive computer processor time requirements, or limited applicability.

Horwitz (Ref 6) developed a method whereby the mode eigenvalue equation for an unstable strip resonator, modified from the original, developed by Fox and Li (Ref 4), was simplified by using first a series of functions found through asymptotic analysis to approximate the field in the resonator, and then the method of stationary phase to approximate the integral. Butts and Avizonis (Ref 2) clarified this approach and modified it to allow consideration of a resonator with circular mirrors. However, neither allowed for the inclusion of a gain medium in the cavity.

Objectives

The objective of this thesis is to develop a computer code allowing analysis of modes in an unstable resonator and to then utilize that code in performing said analysis. The code is to be developed for a strip resonator and account for both bare and loaded cavity cases.

Assumptions

To facilitate modeling of the unstable resonator, certain simplifying assumptions will be made:

1. Scalar diffraction theory will be used to describe the physical situation in the resonator. This is reasonable,

since the dimensions of laser resonators are large compared to optical wavelengths.

2. The Fresnel approximation to the Kirchoff-Fresnel formula is valid. Resonator cavity lengths make this an acceptable assumption.

3. In a Cartesian system, diffraction integrals and mode eigenfunctions are separable. This allows a 1-D strip resonator to be utilized in the following development.

4. One of the resonator mirrors is very much larger than the beam width on that mirror. In other words, that the height of this mirror be considered infinite. This is not an impossible physical constraint.

5. The modes in the strip resonator consist of a fundamental cylindrical wave modified by a finite number of diffraction effects. This assumption is supported by early analysis of unstable resonators. (Ref 9:279)

Procedure

This thesis will start with the Kirchoff-Fresnel diffraction formula and develop, following Horwitz (Ref 6:1529), the eigenvalue equation for a strip resonator

$$\lambda g(x) = \sqrt{\frac{it}{\pi}} \int_{-1}^{1} e^{-it\left(y-\frac{x}{m}\right)^2} g(y)\, dy \qquad (1.3.1)$$

where the eigenfunctions $g(x)$ are the weighting functions of the basic cylindrical wave assumed to be present in the resonator, that is, if the total field is described by $u(x)$ then (Ref 11),

$$u(x) = g(x)e^{-i\pi N_f x^2} \qquad (1.3.2)$$

$e^{-i\pi N_f x^2}$ being the phase curvature term of the basis cylindrical wave.

The eigenvalues will be found by developing a suitable relation from the eigenvalue equation. The total field in the cavity is first assumed to consist of a unit amplitude cylindrical wave plus a finite series of diffraction supplements. This is stated in terms of $g(x)$ as (Ref 2)

$$g(x) = 1 + \sum_{n=1}^{N} c_n H_n(x) \qquad (1.3.3)$$

This expression is substituted in the eigenvalue equation and then an approximation to the integral is developed using a first approximation to the method of stationary phase. The resulting relation will allow the eigenvalues to be expressed as roots of a polynomial with determinable coefficients. The roots can be found by using a general root-finding routine.

An eigenfunction expression may then be found by using

the original assumption ie., equation (1.3.3). However, inherent limitations of the first stationary phase approximation confining applicable x values in this relation require the development of a better approximation to the integral. The higher order expression will be developed, using the higher order approximation to the method of stationary phase (Ref 1), enabling the evaluation of the eigenfunctions throughout a continuous range of x values.

This thesis will then seek to modify the bare cavity expressions to account for a gain medium in the resonator by introducing a gain factor, $e^{2\bar{g}L}$, into the integral abd by relaxing the unit amplitude requirement on the fundamental cylindrical wave.

Organization

The derivation of the basic resonator eigenvalue equation will be covered in Chapter II. Chapter III will present the two applications of that equation: calculation of eigenvalues and evaluation of eigenfunctions. Inclusion of gain considerations will be covered in Chapter IV and Chapter V will contain results of the computer code. Chapter VI will include conclusions and further recommendations.

II. Development of the Eigenvalue Equation

Chapter II addresses the problem of applying the Kirchoff-Fresnel diffraction formula to the desired case of an unstable optical resonator. The development follows that in Reference 6.

A steady state mode will exist in a resonator when the field value on one mirror resulting from one round trip through the resonator multiplied by some complex constant is equal to the original field value on that mirror. Mathematically this can be stated as

$$\gamma E'(x,y) = E(x,y) \qquad 2.1.1$$

where E is the original field distribution on M_2, the second mirror, E' is the distribution after one round trip, and γ is the constant, in general complex.

Wave propagation through the resonator can be expressed using scalar diffraction theory. Wave propagation from a rectangular aperture, dimensions $2a \times 2c$, on one plane to another plane a distance L away, as seen in Fig.1, is given in the Fresnel approximation by

$$E(x_2,y_2) = \frac{ie^{-ikL}}{\lambda L} \int_{-c}^{c}\int_{-a}^{a} E(x_1,y_1)e^{-\frac{ik}{2L}[(x_1-x_2)^2+(y_1-y_2)^2]}dx_1dy_1 \qquad 2.1.2$$

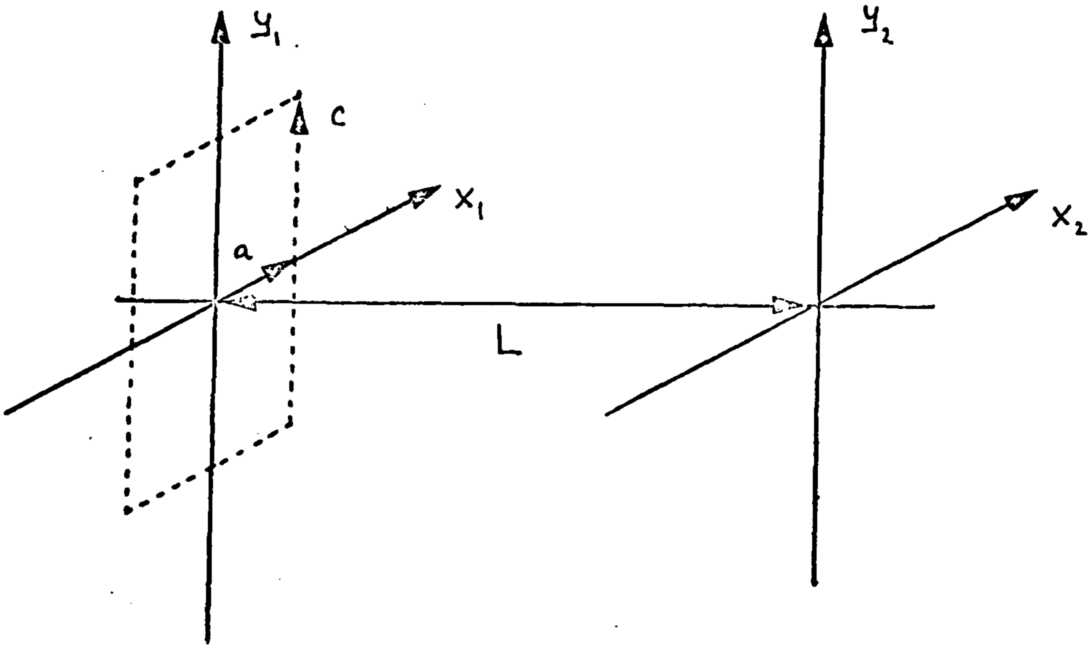


Figure 1

Since this is presented in a Cartesian system, the field distribution can be separated. This is done by assuming (Ref 4:485-486),

$$E(x,y) = U(x)U(y) \qquad 2.1.3$$

Substitution of 2.1.3 into the diffraction formula yields two independent diffraction formulae.

$$U(x_2) = \sqrt{\frac{i}{\lambda L}}\; e^{-\frac{ikL}{2}} \int_{-a}^{a} U(x_1) e^{-\frac{ik}{2L}(x_1 - x_2)^2} dx_1 \qquad 2.1.4$$

$$U(y_2) = \sqrt{\frac{i}{\lambda L}}\, e^{-\frac{ikL}{2}} \int_{-c}^{c} U(y_1) e^{-\frac{ik}{2L}(y_1-y_2)^2} dy_1 \qquad 2.1.5$$

Consideration of only one of these formulae is equivalent to considering diffraction from a strip aperture. No generality is lost, however, since the effects of a finite aperture can be found from the product of two separate strip cases. Thus the one remaining equation is

$$U(x_2) = \sqrt{\frac{i}{\lambda L}}\, e^{-\frac{ikL}{2}} \int_{-a}^{a} e^{-\frac{ik}{2L}(x_1-x_2)^2} U(x_1) dx_1 \qquad 2.1.6$$

Equation 2.1.6 represents propagation from one plane to another. For this to correctly represent propagation in a resonator, the phase lag introduced by mirror curvature must be accounted for.

The phase lag introduced by the mirrors can be expressed as a function of distance from the optic axis. This expression can be derived from the paraxial lens thickness function, (Ref 5:80), which is

$$\Delta(x,y) = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad 2.1.7$$

Where Δ= thickness, Δ_0= maximum lens thickness, x and y are coordinates of the point where the ray of interest is incident on the lens, and R_1 and R_2 are the radii of

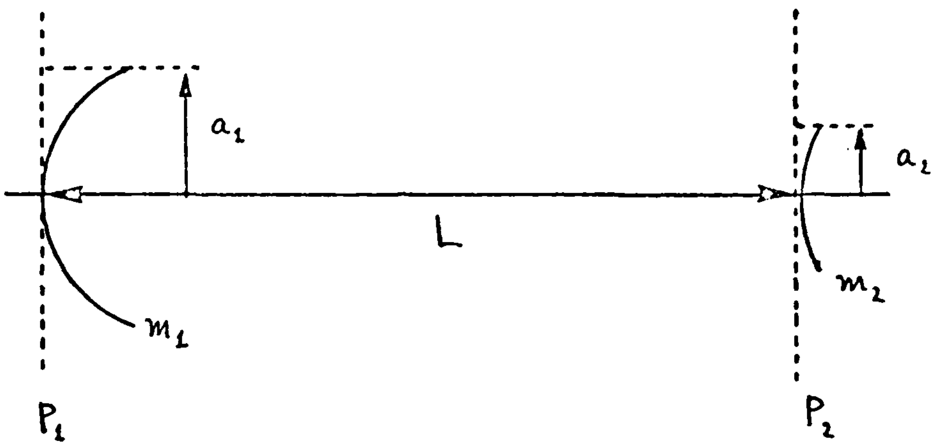


Figure 2

curvature of the lens' surface. In the case of a strip mirror the lens equation applys if

$$R_2 = \infty$$

$$y = 0$$

$$\Delta_0 = 0$$

giving

$$\Delta(x) = \frac{x^2}{2R_i} \qquad 2.1.8$$

for the ith mirror. The phase lag at some particular distance from the optic axis x is given by

$$\Delta\phi(x) = \frac{kx^2}{2R} \qquad 2.1.9$$

However, since

$$g_i = 1 - \frac{L}{R_i} \qquad 2.1.10$$

it is seen that

$$1-g_i = \frac{L}{R_i}$$

and

$$\frac{1}{R_i} = \frac{1-g_i}{L} \qquad 2.1.11$$

where g_i is the g parameter for the i^{th} mirror. Therefore the phase lag can be expressed as

$$\Delta\phi_i = \frac{kx^2}{2}\,\frac{1-g_i}{L} \qquad 2.1.12$$

Introduction of this phase lag into the diffraction formula gives

$$U(x_2) = \sqrt{\frac{i}{\lambda L}}\, e^{-\frac{ikL}{2}} \int_{-a}^{a} U(x_1)$$

$$e^{-\frac{ik}{2L}\left[x_1{}^2+x_2{}^2-2x_1x_2-x_1{}^2(1-g_1)-x_2{}^2(1-g_2)\right]} dx_1$$

$$U(x_2) = \sqrt{\frac{i}{\lambda L}}\, e^{-\frac{ikL}{2}} \int_{-a}^{a} U(x_1) e^{-\frac{ik}{2L}[g_1 x_1{}^2 + g_2 x_2{}^2 - 2x_1 x_2]} dx_1$$

2.1.13

This expression, now modified to describe propagation of $U(x_1)$ from M_1 to M_2, can be used to set up two equations: one for propagation from M_2 to M_1 and the other for propagation from M_1 to M_2. Combination of the two will then yield an expression describing propagation of a field through one round trip in the resonator. The one way formulae are

$$U(x_2) = e^{-\frac{ikL}{2}} \sqrt{\frac{i}{\lambda L}} \int_{-a_1}^{a_1} U(x_1') e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2{}^2 - 2x_1' x_2]} dx_1'$$

2.1.14

and

$$U(x_1) = e^{-\frac{ikL}{2}} \sqrt{\frac{i}{\lambda L}} \int_{-a_2}^{a_2} U(x_2') e^{-\frac{ik}{2L}[g_1 x_1{}^2 + g_2 x_2'^2 - 2x_1 x_2']} dx_2'$$

2.1.15

If 2.1.15 is substituted for $u(x_1')$ in 2.1.14, the resultant expression will give the field on M_2 due to the propagation of an original field on M_2 through one round trip in the resonator. Substitution gives

$$U(x_2) = e^{-ikL} \sqrt{i/\lambda L} \int_{-a_2}^{a_2} \sqrt{i/\lambda L} \int_{-a_1}^{a_1} U(x_2')$$

$$e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2'^2 - 2x_1' x_2']} dx_2' e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2^2 - 2x_1' x_2]} dx_1'$$

2.1.16

In the case considered in Fig. 2, the assumption that M_1 is much bigger than the beam width on that mirror for any laser mode that is likely to resonate, allows a_1 to be thought of as essentially infinite. Then 2.1.16 becomes

$$U(x_2) = e^{ikL} \int_{-a_2}^{a_2} \int_{-\infty}^{\infty} \frac{i}{\lambda L} e^{\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2'^2 - 2x_1' x_2']}$$

$$e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2^2 - 2x_1' x_2]} U(x_2') dx_1' dx_2' \qquad 2.1.17$$

This expression can be simplified by extracting the interior integral

$$\frac{i}{\lambda L} \int_{-\infty}^{\infty} e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2'^2 - 2x_1' x_2']}$$

$$e^{-\frac{ik}{2L}[g_1 x_1'^2 + g_2 x_2^2 - 2x_1' x_2]} dx_1' \qquad 2.1.18$$

Evaluation of this integral in Appendix B yields

$$\sqrt{\frac{i}{2L\lambda g}}\quad e^{-\frac{i}{2\lambda L g_1}[(2g_1g_2-1)(x_2'^2+x_2^2)-2x_2x_2']} \qquad 2.1.19$$

Substitution of this for the complete kernel in 2.1.17 in turn yields

$$U(x_2) = e^{-ikL}\sqrt{\frac{i}{2L\lambda g_1}}\int_{-a_2}^{a_2}$$

$$e^{-\frac{i\pi}{\lambda L 2g_1}[(2g_1g_2-1)(x_2'^2+x_2^2)-2x_2x_2']}\,U(x_2')\;dx_2'$$

2.1.20

To simplify this further, the definitions

$$2g_1g_2-1 \equiv g \qquad 2.1.21$$

and

$$\frac{a_2{}^2}{2g_1\lambda L} = \frac{F_2}{2g_1} \equiv F \qquad 2.1.22$$

are introduced.

Here, g_1 and g_2 are the familiar g parameters and F_2 is the ordinary Fresnel number of the smaller feedback mirror. The ordinary Fresnel number is defined as the additional length per pass in half wavelengths for a ray traveling from one mirror's center to the other mirror's edge, compared to one traveling from mirror center to mirror

center. (Ref 11:159-161).

The dimensions of the quantities are also scaled such that $a_2=1$. These modifications yield

$$U(x_2) = e^{-ikL}\sqrt{iF}\int_{-1}^{1} U(x_2')e^{-i\pi F[g(x_2'^2+x_2)-2x_2x_2']}dx_2' \qquad 2.1.23$$

Imposition of the reproducobility constraint, equation 2.1.1, and absorption of the constant e^{-ikL} into γ yields

$$\gamma U(x_2) = \sqrt{iF}\int_{-1}^{1} U(x_2')e^{-i\pi F[g(x_2'^2+x_2^2)-2x_2x_2']}dx_2' \qquad 2.1.24$$

Introducing the dummy variable y and dropping subscripts and superscripts yields

$$\gamma U(x) = \sqrt{iF}\int_{-1}^{1} U(y)e^{-i\pi F[g(x^2+y^2)-2xy]}dy \qquad 2.1.25$$

To further simplify this equation, the following quantities are defined

$$N_f = \frac{F}{2}(m - \frac{1}{m}) \qquad 2.1.26$$

and $g(x)$ such that

$$U(x) = e^{-i\pi N_f x^2} g(x) \qquad 2.1.27$$

N_f is the equivalent Fresnel number of the resonator. The equivalent Fresnel number can be interpreted as the additional path length per pass in half wavelengths for a ray traveling from a mirror's virtual center to the edge of the next mirror, as opposed to a ray traveling from the virtual center of one mirror to the actual center of the next (Ref 11:159-161). The virtual center is defined as that point from which a cylindrical wave would eminate if that wave were to be reflected from a feedback mirror, and then return to the original mirror in the same form as when it left (Ref 9:279-280). That cylindrical wave is then assumed to take the form

$$e^{-i\pi N_f x^2} \qquad 2.1.28$$

and the entire wave function is assumed to be based on that wave, stated by 2.1.27. Substitution of 2.1.26 and 2.1.27 into 2.1.25 yields

$$\gamma g(x) e^{-i\pi \frac{F}{2}\left(\frac{m^2-1}{m}\right) x^2} = \sqrt{iF} \int_{-1}^{1} g(y) e^{-i\pi F y^2 \left(\frac{m^2-1}{2m}\right)}$$

$$e^{-i\pi F[g(x^2+y^2)-2xy]}\, dy \qquad 2.1.29$$

After some manipulation, detailed in Appendix C,

2.1.29 simplifies to the final form of the resonator integral equation

$$\gamma g(x) = \sqrt{it/\pi m} \int_{-1}^{1} g(y) e^{-it(y-\frac{x}{m})^2} dy \qquad 2.1.30$$

where

$$t = \pi m F \qquad 2.1.31$$

and if

$$\lambda = \gamma\sqrt{m} \qquad 2.1.32$$

$$\lambda g(x) = \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} g(y)\, dy \qquad 2.1.33$$

III. Determination of Eigenvalues and Evaluation of Eigenfunctions

Chapter III is concerned with solving the resonator mode eigenvalue equation and with developing expressions for the resulting eigenfunctions. The eigenfunctions are most desireable since they will ultimately express field values across the output mirror plane.

Approximation of Eigenvalue Equation

The eigenvalue equation that must be solved is

$$\lambda g(x) = \sqrt{it/\pi} \int_{-1}^{1} g(y) e^{-it(y-\frac{x}{m})^2} dy \qquad 3.1.1$$

where $g(x)$ is the quantity multiplying the primary cylindrical wave expressed as a function of normal distance from the optic axis.

Now it is assumed that the field on the mirror before the round trip, $U(y)$, consists of a unit amplitude cylindrical wave plus an infinite series of edge diffracted waves given by some functions $H_n(y)$ (Ref 2). In terms of $g(y)$ this is stated as

$$g(y) = 1 + \sum_{n=1}^{\infty} c_n H_n(y) \qquad 3.1.2$$

The physical basis for this assumption is that the original field on M_2 will consist of that primary cylindrical wave which makes the round trip unchanged plus other contributions which are the diffraction additions to that wave from previous reflections. To make this viable, however, it is then assumed that the series terminates when eventually some function $H_N(y)$ is the last contribution that has any new effect on the field, or that $H_{N+1}(y)$ is constant. If the resonator is thought of as an infinite lens train, the mode components between the last two lenses will consist of the basic cylindrical wave and one diffraction effected wave from each preceeding lens group. The series terminates when the consideration of another lens group, farther back, adds no more new information to the final mode. Then the addition of one more diffraction effected wave would add only to amplitude, and not change the shape of the total wave. 3.1.2 then becomes $g(y)=1+\sum_{n=1}^{N} c_n H_n(y)$. A good approximation is to let (Ref. 6:1533)

$$N \geq \frac{\ln 250 N_f}{\ln m} \qquad 3.1.3$$

and the quality of this approximation is displayed in Appendix E.

When 3.1.2 is substituted into 3.1.1, the result is

$$\lambda g(x) = \sqrt{it/\pi} \int_{-1}^{1} \left\{ 1+\sum_{n=1}^{N} c_n H_n(y) \right\} e^{-it(y-\frac{x}{m})^2} dy \qquad 3.1.4$$

Some method of approximating this integral is needed. The method chosen is the method of stationary phase. This method states that an integral of the form

$$\int_a^b e^{-itp(y)} q(y)\, dy \qquad 3.1.5$$

can, when t is large and $q(y)$ is slowly varying, be expressed as a series, the first two terms of which are approximately (Ref.2:1073).

$$e^{-i\pi/4} q(y_0) e^{-itp(y_0)} \sqrt{\frac{2\pi}{tp''(y_0)}}$$

$$+ \frac{i}{t}\left[\frac{q(b)}{p'(b)}\, e^{-itp(b)} - \frac{q(a)}{p'(a)}\, e^{-itp(a)}\right] \qquad 3.1.6$$

where y_0 is the point of stationary phase, ie.

$$p'(y_0) = 0 \qquad 3.1.7$$

To utilize this however some explicit form of $H_n(y)$ is needed. The form used here is the same as that developed by Horwitz through asymptotic analysis of the resonator integral, 2.1.33. The form is as follows:

Given the functions (Ref 3)

$$F(x,t) = -\frac{1}{2\sqrt{i\pi t}}\,\frac{e^{-it(1-x)^2}}{1-x} \qquad 3.1.8$$

$$G(x,t) = -\frac{1}{2\sqrt{i\pi t}}\frac{e^{-it(1+x)^2}}{1+x} \qquad 3.1.9$$

the functions $F_n(x)$ and $G_n(x)$ are formed such that

$$F_n(x) = F\left(\frac{x}{m^n}, \frac{t}{m_{n-1}}\right) \qquad 3.1.10$$

and

$$G_n(x) = G\left(\frac{x}{m^n}, \frac{t}{m_{n-1}}\right) \qquad 3.1.11$$

where

$$m_n = \sum_{k=0}^{n} m^{-2k} \qquad 3.1.12$$

and m is the magnification.

It is therefore seen that

$$F_n(x) = -\frac{\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\frac{e^{-it(1-\frac{x}{m^n})^2/m_{n-1}}}{1-x/m^n} \qquad 3.1.13$$

and

$$G_n(x) = -\frac{\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\frac{e^{-it(1+\frac{x}{m^n})^2/m_{n-1}}}{1+x/m^n} \qquad 3.1.14$$

$H_n(x)$ is then assumed to be some combination of these functions:

$$c_n H_n(x) = a_n F_n(x) + b_n G_n(x) \qquad 3.1.15$$

Since the cavity under consideration here is centered on the optic axis, symmetry dictates either odd or even field functions. To get an even field function, then it is assumed that $H_n(x)$ is *even*. *Odd would require that* $H_n(x)$ be odd.

It is seen from 3.13-3.15 that $H_n(x)$ can be made even if $a_n = b_n$. So, if

$$a_n = b_n \qquad 3.1.16$$

then with

$$c_n = a_n = b_n \qquad 3.1.17$$

$$c_n H_n(x) = c_n(F_n(x) + G_n(x)) \qquad 3.1.18$$

and

$$c_n H_n(-x) = c_n(F_n(-x) + G_n(-x)) \qquad 3.1.19$$

However, since

$$F_n(x) = G_n(-x) \qquad 3.1.20$$

$$c_n H_n(-x) = c_n(G_n(x) + F_n(x)) \qquad 3.1.21$$

$$= c_n H_n(x) \qquad 3.1.22$$

Thus, $H_n(x)$ is an even function.

Similarly, $H_n(x)$ can be made odd by assuming

$$a_n = -b_n \qquad 3.1.23$$

and that

$$a_n = -b_n = c_n \qquad 3.1.24$$

it is seen that

$$c_n H_n(x) = c_n(F_n(x)-G_n(x)) \qquad 3.1.25$$

and

$$c_n H_n(-x) = c_n(F_n(-x)-G_n(-x) \qquad 3.1.26$$

$$= c_n(G_n(x)-F_n(x)) \qquad 3.1.27$$

$$= -c_n H_n(x) \qquad 3.1.28$$

Thus $H_n(x)$ is an odd function. One additional assumption is that in the odd case, the amplitude of the cylindrical wave is zero. This is necessary for the field function to be odd.

In the following development, the even parity case will be the one dealt with. The odd parity equations can be found from those for the even case by deleting the leading term in eq. 3.1.2 and following the procedure as above.

Therefore, the eigenvalue equation to be solved is

$$\lambda g(x) = \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2}$$

$$\left\{1+\sum_{n=1}^{N}(a_n F_n(y)+b_n G_n(y))\right\} dy \qquad 3.1.29$$

Substitution of the actual forms of the functions allows explicit forms of $p(y)$ and $g(y)$ found in 3.1.4 and 3.1.5 to be found. Employing equation 3.1.5, according to Appendix A allows the following first order approximation

$$\lambda g(x) \simeq 1+F_1(x)+G_1(x)$$

$$+\sum_{n=1}^{N}(a_n F_{n+1}(x)+b_n G_{n+1}(x))$$

$$+F_1(x)\sum_{n=1}^{N}(a_n F_n(1)+b_n G_n(1))$$

$$+G_1(x)\sum_{n=1}^{N}(a_n F_n(-1)+b_n G_n(-1)) \qquad 3.1.30$$

$$\simeq 1+H_1(x)+\sum_{n=1}^{N}c_n H_{n+1}(x)+H_1(x)\sum_{n=1}^{N}c_n H_n(1) \qquad 3.1.31$$

$$= \lambda(1+\sum_{n=1}^{N}c_n H_n(x)) \qquad 3.1.32$$

In the odd case this would become

$$\lambda \sum_{n=1}^{N} c_n H_n(x) \simeq \sum_{n=1}^{N} c_n H_{n+1}(x) + H_1(x) \sum_{n=1}^{N} c_n H_n(1) \qquad 3.1.33$$

The Eigenvalue Polynomial

In the even parity case, the eigenvalue equation is approximately given by

$$\lambda\left(1+\sum_{n=1}^{N} c_n H_n(x)\right) \simeq 1 + H_1(x) + \sum_{n=1}^{N} c_n H_{n=1}(x)$$

$$+H_1(x) \sum_{n=1}^{N} c_n H_n(1) \qquad 3.2.1$$

The relation between λ and the known functions is constructed in the following manner:

First, the coefficients of terms in 3.2.1 involving $H_n(x)$, where $n \neq 1$, are set equal:

$$\lambda c_{n+1} = c_n \qquad 3.2.2$$

It follows that

$$c_{n+1} = \frac{c_n}{\lambda} \qquad 3.2.3$$

$$c_2 = \frac{c_1}{\lambda}, \quad c_3 = \frac{c_2}{\lambda} = \frac{c_1}{\lambda \cdot \lambda} \qquad 3.2.4$$

and generally it is seen that

$$c_{n+1} = \frac{c_1}{\lambda^n} = \frac{c_n}{\lambda} \qquad 3.2.5$$

In other words

$$c_1\lambda = c_n\lambda^n \qquad 3.2.6$$

$$= c_N\lambda^N \qquad 3.2.7$$

This in turn implies that

$$c_n = c_N\lambda^{N-n} \qquad 3.2.8$$

Equating coefficients of $H(x)$ now yields

$$\lambda c_1 = 1+\sum_{n=1}^{N} c_n H_n(1) \qquad 3.2.9$$

Substituting for c_n and c_1 according to 3.2.8, gives

$$\lambda c_N\lambda^{N-1} = 1+\sum_{n=1}^{N} c_N\lambda^{N-n}H_n(1) \qquad 3.2.10$$

Equating constant terms in 3.2.1 shows that

$$\lambda = 1+c_N H_{N+1} \qquad 3.2.11$$

$$\lambda - 1 = c_N H_{N+1} \qquad 3.2.12$$

$$\frac{\lambda - 1}{H_{N+1}} = c_N \qquad 3.2.13$$

Substituting for c_N in 3.2.10 according to 3.2.13 yields

$$\frac{\lambda(\lambda-1)\lambda^{N-1}}{H_{N+1}} = 1 + \sum_{n=1}^{N} \frac{(\lambda-1)\lambda^{N-n}}{H_{N+1}} H_n(1) \qquad 3.2.14$$

or

$$\lambda^N(\lambda-1) = H_{N+1} + (\lambda-1) \sum_{n=1}^{N} \lambda^{N-n} H_n(1) \qquad 3.2.15$$

which is a *polynomial in the complex variable* λ. Its roots can be determined from any root-finding subroutine, since its coefficients all involve known quantities such as

$$H_n(1)$$

or the constant

$$H_{N+1}$$

It is from this polynomial that the mode eigenvalues of the resonator are determined. A preliminary evaluation of the eigenfunction for a particular mode can be made by substituting into equation 3.1.2 the values for c_n, which are given by

$$c_n = c_N \lambda^{N-n}$$

$$= \frac{(\lambda-1)}{H_{N+1}} \lambda^{N-n} \qquad 3.2.16$$

However, due to the singularities in the first approximation to the integral, 3.1.6, whenever x approaches y_0 , this particular expression for the eigenfunction is invalid. This problem will be remedied in the next section.

The odd parity solution is given by 3.1.33, and the polynomial development for that case is as follows.

After equating the coefficients of $H_n(x)$, $n \neq 1$, it is seen that the same relations arise as 3.2.2 - 3.2.8.

Equating coefficients of $H_1(x)$ indicates that

$$\lambda c_1 = \sum_{n=1}^{N} c_n H_n(1) \qquad 3.2.17$$

Equating constant terms indicates that

$$0 = c_N H_{N+1}$$

This is only reasonable since the condition imposed on N , namely that H_{N+1} is a constant, also implies that

$$F_{N+1} = \text{Constant} = G_{N+1} \qquad 3.2.18$$

and since

$$H_{N+1} = F_{N+1} - G_{N+1} \qquad 3.2.19$$

$$H_{N+1} = 0 \qquad 3.2.20$$

This indicates that c_n is completely arbitrary since there are no other restrictions imposed by either 3.2.8 or 3.2.17. If c_n is indeed arbitrary, and

$$c_n = c_N \lambda^{N-n} \qquad 3.2.21$$

then c_n can be chosen such that

$$c_n \lambda^n = c_N \lambda^N = 1 \qquad 3.2.22$$

leaving the relation

$$c_n = \lambda^{-n} \qquad 3.2.23$$

which can be used in the limited range eigenfunction expression for the odd parity case.

The polynomial is developed by substitution for c_n of 3.2.8 in 3.2.17 giving

$$\lambda c_1 = \sum_{n=1}^{N} c_n H_n(1) \qquad 3.2.24$$

$$\lambda c_N \lambda^{N-1} = \sum_{n=1}^{N} c_N \lambda^{N-n} H_n(1) \qquad 3.2.25$$

$$\lambda^N = \sum_{n=1}^{N} \lambda^{N-n} H_n(1) \qquad 3.2.26$$

Development of Eigenfunction Expressions Valid for All X

To develop an eigenfunction expression valid for all x, it is first necessary to return to the original equation, 2.1.32, which is

$$\lambda g(x) = \sqrt{\frac{it}{\pi}} \int_{-1}^{1} g(y) e^{-it(y-\frac{x}{m})^2} dy \qquad 3.3.1$$

Since the eigenvalues are known or can be determined, it can then be said that

$$g(x) = \frac{1}{\lambda}\sqrt{it/\pi} \int_{-1}^{1} \left[1+\sum_{n=1}^{N} c_n H_n(y)\right] e^{-it(y-\frac{x}{m})^2} dy \qquad 3.3.2$$

One might question the validity of this expression, since λ was determined from the first order approximation. However, that previous approximation yields perfectly valid values for λ, because all that determines the mode eigenvalue is the field on the smaller feedback mirror,

where $x \leq 1$. In this region, the approximation is always valid. Therefore the λ's are perfectly valid.

All expressions and quantities on the right side of 3.3.1 are known, and therefore one can once again utilize the method of stationary phase, but in a second approximation, yielding an expression no longer as simple as 3.1.6 but one that is valid for all x . The higher approximation to the integral is given by (Ref 1)

$$I \approx e^{-itp(b)} q(b) \sqrt{\frac{\pi}{tp''(b)}} \; e^{-\frac{itp'(b)^2}{2p''(b)}}$$

$$\left\{ E^*\left(\sqrt{\frac{t}{\pi p''(b)}} \; p'(b)\right) - \frac{1-i}{2} \right\} - e^{-itp(a)}$$

$$q(a) \sqrt{\frac{\pi}{tp''(a)}} \; e^{-\frac{itp'(a)^2}{2p''(a)}}$$

$$\left\{ E^*\left(\sqrt{\frac{t}{\pi p''(a)}} \; p'(a)\right) - \frac{1-i}{2} \right\} \qquad 3.3.3$$

wnen y_0 is such that

$$y_0 \leq a \qquad 3.3.4$$

and

$$\simeq e^{-i\pi/4} q(y_0)\, e^{-itp(y_0)} \sqrt{\frac{2\pi}{tp''(y_0)}}$$

$$+e^{-itp(b)} \sqrt{\frac{\pi}{tp''(b)}}\; e^{-\frac{itp'(b)^2}{2p''(b)}} \left\{E^*\left(\sqrt{\frac{t}{\pi p''(b)}}\; p'(b)\right) - \frac{1-i}{2}\right\}$$

$$-e^{-tp(a)} \sqrt{\frac{\pi}{tp''(a)}}\; e^{-\frac{itp'(a)^2}{2p''(a)}} \left\{E^*\left(\sqrt{\frac{t}{\pi p''(a)}}\; p'(a)\right) + \frac{1-i}{2}\right\}$$

3.3.5

when y_0 is such that

$$a \leq y \leq b \qquad 3.3.6$$

and

$$\simeq e^{-itp(b)} \sqrt{\frac{\pi}{tp''(b)}}\; e^{-\frac{itp'(b)^2}{2p''(b)}} \left\{E^*\left(\sqrt{\frac{t}{\pi p''(b)}}\; p'(b)\right) + \frac{1-i}{2}\right\}$$

$$-e^{-itp(a)} \sqrt{\frac{\pi}{tp''(a)}}\; e^{-\frac{itp'(a)^2}{2p''(a)}} \left\{E^*\left(\sqrt{\frac{t}{\pi p''(a)}}\; p'(a)\right) + \frac{1-i}{2}\right\}$$

3.3.7

when y_0 is such that

$$y_0 \geq b \qquad 3.3.8$$

Here, E* is the complex conjugate of the Fresnel integral.

In both the even and the odd cases the integral

$$\frac{1}{\lambda}\sqrt{it/\pi}\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} \sum_{n=1}^{N}(a_nF_n(y)+b_nG_n(y)) \quad dy \qquad 3.3.9$$

must be evaluated. Specific differences for even and odd cases will be treated later. Manipulating 3.3.5 yields

$$\frac{1}{\lambda}\sqrt{it/\pi}\sum_{n=1}^{N}\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} (a_nF_n(y)+b_nG_n(y)) \; dy \qquad 3.3.10$$

To make use of equations 3.3.3, 3.3.5 and 3.3.7, it is necessary to get p(y) and q(y) expressions for the n^{th} term in the series. Substitution of the explicit forms of the F_n and G_n functions yields an integral of the form

$$\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2}\left[-\frac{\sqrt{a_n\; m_{n-1}}}{2\sqrt{i\pi t}}\;\frac{e^{-it(1-\frac{y}{m^n})^2/m_{n-1}}}{1-\frac{y}{m^n}}\right.$$

$$\left.-\frac{b_n\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\;\frac{e^{-it(1+y/m^n)^2/m_{n-1}}}{1+\frac{y}{m^n}}\right] dy \qquad 3.3.11$$

where m_n has been previously defined in 3.1.12.

Upon consideration of the term involving the a_n constants, it is seen that

$$p(y) = (y-\frac{x}{m})^2 + \frac{(1-\frac{y}{m^n})^2}{m_{n-1}} \qquad 3.3.12$$

$$p'(y) = 2(y-\frac{x}{m}) - \frac{2}{m_{n-1}}\frac{1}{m^n}(1-\frac{y}{m^n}) \qquad 3.3.13$$

$$p''(y) = 2 + \frac{2}{m_{n-1}m^{2n}} \qquad 3.3.14$$

$$q(y) = \frac{1}{1-\frac{y}{m^n}} \qquad 3.3.15$$

Solving for y_0 yields

$$y_0 - \frac{x}{m} - \frac{1}{m_{n-1}m^n} + \frac{y_0}{m_{n-1}m^{2n}} = 0 \qquad 3.3.16$$

$$y_0(1+\frac{1}{m_{n-1}m^n}) = \frac{x}{m} + \frac{1}{m_{n-1}m^n} \qquad 3.3.17$$

$$y_0^a = \left(\frac{x}{m} + \frac{1}{m^n m_{n-1}}\right)\left(\frac{1}{1+\frac{1}{m_{n-1}m^{2n}}}\right) \qquad 3.3.18$$

Similarly, for the part involving the b_n constants it is seen that

$$p(y) = \left(y-\frac{x}{m}\right)^2 + \frac{\left(1 + \frac{y}{m^n}\right)^2}{m_{n-1}} \qquad 3.3.19$$

$$p'(y) = 2\left(y-\frac{x}{m}\right) + \frac{2}{n-1}\,\frac{1}{m^n}\left(1+\frac{y}{m^n}\right) \qquad 3.3.20$$

$$p''(y) = 2 + \frac{2}{m^{2n}m_{n-1}} \qquad 3.3.21$$

$$q(y) = \frac{1}{1 + \frac{y}{m^n}} \qquad 3.3.22$$

Solving for y_0 , it is seen that

$$y_0 - \frac{x}{M} + \frac{1}{m^n m_{n-1}} + \frac{y_0}{m^{2n}m_{n-1}} = 0 \qquad 3.3.23$$

$$y_0\left(1 + \frac{1}{m^{2n}m_{n-1}}\right) = \frac{x}{m} - \frac{1}{m^n m_{n-1}} \qquad 3.3.24$$

$$y_0^b = \left(\frac{x}{m} - \frac{1}{m^n m_{n-1}}\right)\left(\frac{1}{1 + \frac{1}{m^{2n}m_{n-1}}}\right) \qquad 3.3.25$$

These expressions can now be substituted into the overall approximations to the integral. However, in evaluation careful consideration of the y_0 values must be taken, in

order that the proper form of the approximation is used. This is rather complicated, since there are two y_0's : one for the a_n term, and one for the b_n term.

In order to simplify things, let

$$
-\frac{a_n\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\left\{\frac{e^{-it\left[(1-\frac{x}{m})^2+(1-\frac{1}{m^n})^2/m_{n-1}\right]}}{1-\frac{1}{m^n}}\sqrt{\frac{\pi}{2t\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right.
$$

$$
\cdot e^{+4it\left[1-\frac{x}{m}-\frac{1-\frac{1}{m^n}}{m^n m_{n-1}}\right]^2\Big/4\cdot 1+\frac{1}{m^{2n}m_{n-1}}}\cdot\left[E^*\left(\sqrt{\frac{t}{2\pi\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right.\right.
$$

$$
\left.\left.\left.\cdot\left(2(1-\frac{x}{m})-\frac{2}{m^n m_{n-1}}\left(1-\frac{1}{m^n}\right)\right)\right)-\frac{1-i}{2}\right]\right\}+\frac{a_n\ m_{n-1}}{2\sqrt{i\pi t}}
$$

$$
\left\{e^{-it\left[(-1-\frac{x}{m})^2+(1+\frac{1}{m^n})^2/m_{n-1}\right]}\sqrt{\frac{\pi}{2t\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right.
$$

$$
e^{4it\left[-1-\frac{x}{m}-(-1+\frac{1}{m^n})\ m^n m_{n-1}\right]^2\Big/4\cdot 1+1/m^{2n}m_{n-1}}
$$

$$
\left.\left[E^*\sqrt{\frac{t}{2\pi\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\left(2(-1-\frac{x}{m})-\frac{2}{m^n m_{n-1}}\ (1+\frac{1}{m^n})\right)-\frac{1-i}{2}\right]\right\}
$$

$$
= \text{ATERM} \qquad 3.3.26
$$

when $y_0^a \le -1$. When y^b is in the same region, let

$$- \frac{b_n\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\left\{\frac{e^{-it\left[(1-\frac{x}{m})^2+(1+\frac{1}{m^n})^2/m_{n-1}\right]}}{1-\frac{1}{m^n}}\sqrt{\frac{\pi}{2t\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right.$$

$$\cdot e^{4it\left[1-\frac{x}{m}+(1+\frac{1}{m^n})/m_{n-1}m^n\right]^2/4\cdot 1+\frac{1}{m^{2n}m_{n-1}}}$$

$$\left.\left\{E^*\left(\sqrt{\frac{t}{2\pi\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\left(2(1-\frac{x}{m})+\frac{2}{m^n m_{n-1}}(1+\frac{1}{m^n})\right)\right)-\frac{1-i}{2}\right\}\right\}$$

$$+ \frac{b_n\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\left\{\frac{e^{-it\left[(-1-\frac{x}{m})^2+(1-\frac{1}{m^n})^2/m_{n-1}\right]}}{1-\frac{1}{m^n}}\sqrt{\frac{\pi}{2t\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right.$$

$$\cdot e^{4it\ -1-\frac{x}{m}+(1-\frac{1}{m^n})/m_{n-1}m^n\ ^2\ 4\cdot 1+\frac{1}{m^{2n}m_{n-1}}}$$

$$\left.\cdot\left\{E^*\left(\sqrt{\frac{t}{2\pi\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\left(2(-1-\frac{x}{m})+\frac{2}{m^n m_{n-1}}\left(1-\frac{1}{m^n}\right)\right)\right)-\frac{1-i}{2}\right\}\right\}$$

$$= \text{BTERM} \qquad 3.3.27$$

Then, it can be said that

$$\int_{-1}^{1} e^{-it\left(y-\frac{x}{m}\right)^2} (a_n F_n(y)+b_n G_n(y)) \simeq \text{ATERM} + \text{BTERM} \qquad 3.3.28$$

provided that both points of stationary phase are less than negative one, the lower endpoint of the integral. Therefore, in that region, 3.3.10 is equal to

$$\frac{1}{\lambda}\sqrt{it/\pi}\sum_{n=1}^{N}\{\text{ATERM} + \text{BTERM}\} \qquad 3.3.29$$

However, if one or both points of stationary phase fail the magnitude condition, the expressions ATERM and BTERM can be corrected through some slight modifications. If $y_0^a \geq 1$, or $y_0^b \geq -1$ all that has to be done is to change the sign of each $\frac{1-i}{2}$ term. If either y_0 is such that

$$-1 \leq y \leq 1 \qquad 3.3.30$$

then two things must be done. First, only the second $\frac{1-i}{2}$ term requires a change of sign. Second, the stationary phase point contribution term must be added to the entire expression.

For the a_n term, this contribution is

$$-\frac{a_n\sqrt{m_{n-1}}}{2\sqrt{i\pi t}}\left\{\frac{e^{-i\pi/4}}{1-y_0^a/m^n}\, e^{-it\left[\left(y_0^a-\frac{x}{m}\right)^2+\left(1-\frac{y_0^a}{m^n}\right)^2\Big/ m_{n-1}\right]}\sqrt{\frac{\pi}{t\left(1+\frac{1}{m^{2n}m_{n-1}}\right)}}\right\}$$

3.3.31

and for the b_n term the contribution is

$$- \frac{b_n \sqrt{m_{n-1}}}{2\sqrt{i\pi t}} \; \frac{e^{-i\pi/4}}{1 \; y_0^b/m^n} \; e^{-it\left[(y_0^a - \frac{x}{m})^2 + (1+\frac{y_0^b}{m^n})^2 / m_{n-1}\right]} \sqrt{\frac{\pi}{t\left(1+\frac{}{m^{2n} m_{n-1}}\right)}}$$

3.3.32

These expressions are added to ATERM or BTERM, whichever is required. In this way the complete expression for the integral

$$\frac{1}{\lambda} \sum_{n=1}^{N} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} (a_n F_n(y) + b_n G_n(y)) \, dy \qquad 3.3.33$$

can be stated.

However, in the even parity case, one more modification must be made. The term involving the '1' must be added to the expression. Explicity, that term is

$$\frac{1}{\lambda} \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} dy \qquad 3.3.34$$

From this it is seen that

$$p(y) = (y - \frac{x}{m})^2 \qquad 3.3.35$$

$$p'(y) = 2(y - \frac{x}{m}) \qquad 3.3.36$$

$$p''(y) = 2 \qquad 3.3.37$$

$$q(y) = 1 \qquad 3.3.38$$

and

$$y_0 = \frac{x}{m} \qquad 3.3.39$$

Substituting into 3.3.2 for $y_0 \leq -1$ gives

$$\frac{1}{\lambda}\sqrt{it/\pi}\left[e^{-it(1-\frac{x}{m})^2}\, e^{-it/4\cdot 4(1-\frac{x}{m})^2}\left\{E^*\left(\sqrt{\frac{t}{2\pi}}\cdot 2(1-\tfrac{x}{m})\right) - \frac{1-i}{2}\right\}\right.$$

$$\left. -e^{-it(-1-\frac{x}{m})^2}\sqrt{\frac{\pi}{2t}}\, e^{it/4\cdot 4(-1-\frac{x}{m})^2}\left\{E^*\ \frac{t}{2\pi}\cdot 2(1-\tfrac{x}{m}) \quad - \frac{1-i}{2}\right\}\right] \qquad 3.3.40$$

If $y_0 \geq 1$, once again, all that needs to be done is to change the sign of the $\frac{1-i}{2}$ terms. If y_0 is such that 3.3.30 is satisfied, then only the second $\frac{1-i}{2}$ term is changed in sign and the stationary phase point contribution term is added. That term is given by

$$e^{-i\pi/4}\sqrt{\pi/t} \qquad 3.3.41$$

Thus, the higher order approximation expressions for the eigenfunctions

$$\frac{1}{\lambda}\sqrt{it/\pi}\sum_{n=1}^{N}\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2}\left(1+a_nF_n(y)+b_nG_n(y)\right)dy \qquad 3.3.42$$

in the even case, and

$$\frac{1}{\lambda}\sqrt{it/\pi}\sum_{n=1}^{N}\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2}(a_nF_n(y)+b_nG_n(y))\,dy \qquad 3.3.43$$

in the odd case are expressed, per 3.3.2-3.3.4. These expressions are valid for all x . In this way, the fields across the output mirror plane can be evaluated. It follows that intensities are then given by

$$I = E^*E = g^*g \qquad 3.3.44$$

where E is given by the eigenfunctions.

IV. Modifying the Expressions to Account for Gain

Chapter IV addresses the problem of generalizing the previous development so that the expressions can account for the presence of gain in the resonator. It is noted here that the method set forth here is not the only way to include gain in mode analysis (Ref 8).

In this thesis, the method taken to include gain in the preceding development will require two changes in that development. The first is that the fundamental cylindrical wave is no longer assumed to be of unit amplitude. In the even case, to which consideration will be limited, $g(x)$ is then assumed to be of the form

$$g(x) = h + \sum_{n=1}^{N} c_n H_n(x) \tag{4.1.1}$$

where h is the amplitude of the basis wave, in general not equal to 1, which will be determined later. The second modification made is to include a gain factor, $e^{2\bar{g}(y)L}$, in the kernel of the diffraction integral. The integral equation then becomes

$$\lambda g(x) = \sqrt{it/\pi} \int_{-1}^{1} e^{2\bar{g}(y)L} \, e^{-it(y-\frac{x}{m})^2} g(y) \, dy \tag{4.1.2}$$

$$\lambda\left(h+\sum_{n=1}^{N} c_n H_n(x)\right) = \sqrt{it/\pi}\int_{-1}^{1} e^{2\overline{g}(y)L}\, e^{-it(y-\frac{x}{m})^2}\left(h+\sum_{n=1}^{N} c_n H_n(y)\right)dy \qquad 4.1.3$$

The best value of h can be found from a relaxation process wherein gains are assumed to be equal to losses. First however, the equations require a round value as a starting point.

To determine that rough value, it is assumed that there is a uniform intensity across the laser cavity, in particular, at the output plane. Thus, the gain there is affected in a similarly uniform manner. If the gain medium is homogeneous, then

$$\overline{g}(x) = \frac{g_0}{1 + 2\frac{I(x)}{I_{sat}}} = \overline{g} \qquad 4.1.4$$

where g_0 is the small signal gain, and I_{sat} is the saturation intensity, both determinable from actual laser parameters. $I(x)$ is multiplied by 2 since there are intensity contributions from two waves, one propagating in each direction.

If the uniform intensity across the feedback mirror is I_f, then the feedback power is given by

$$P_f = I_f \cdot 2ad \qquad 4.1.5$$

where 2ad is the mirror area.

After one round trip, the power would be

$$P_r = P_f e^{2\bar{g}L} \qquad 4.1.6$$

and the round trip intensity would then be

$$I_r = \frac{P_r}{2mad} \qquad 4.1.7$$

since the area of the beam is now increased by the magnification (neglecting diffraction). In a steady state situation, I_r must equal I_f, and therefore it follows that

$$I_f = I_r = \frac{P_r}{2mad} = \frac{P_f e^{2\bar{g}L}}{2mad} = \frac{I_f 2ade^{2\bar{g}L}}{2mad} \qquad 4.1.8$$

and thus

$$m = e^{2\bar{g}L} \qquad 4.1.9$$

$$2\bar{g}L = \ln m \qquad 4.1.10$$

Substituting 4.1.4 for $\bar{g}$, it is then seen that

$$\ln m = 2L \frac{g_0}{1+2\frac{I(x)}{I_{sat}}} \qquad 4.1.11$$

$$1+2\frac{I(x)}{I_{sat}} = \frac{2g_0L}{\ln m} \qquad 4.1.12$$

$$\frac{I(x)}{I_{sat}} = \left(\frac{2g_0L}{\ln m} - 1\right)\frac{1}{2} \qquad 4.1.13$$

$$= \frac{g_0L}{\ln m} - \frac{1}{2} \qquad 4.1.14$$

This ratio is the ratio of the intensity on the mirror to the saturation intensity, and it will be considered as the relative intensity of the fundamental cylindrical wave. Therefore,

$$h = \sqrt{\frac{g_0L}{\ln m} - \frac{1}{2}} \qquad 4.1.15$$

in a first approximation. This will give a rough starting point for h from which the equations can begin.

Considering the new integral equation, 4.1.3, in light of the first stationary phase approximation, it is seen that a term has been added to the various $q(y)$'s . From the approximation it is then concluded from the results of Appendix A, that

$$\lambda(h+c_nH_n(x)) \simeq he^{2\overline{g}(y_0)L}+he^{2\overline{g}(1)L}H_1(x)+\sum_{n=1}^{N} e^{2\overline{g}(y_0^a)}a_nF_{n+1}(x)$$

$$+\sum_{n=1}^{N} e^{2\overline{g}(y_0^b)}b_nG_{n+1}(x)+F_1(x)\sum_{n=1}^{N} c_nH_n(1)e^{2\overline{g}(1)L}+G_1(x)\sum_{n=1}^{N} c_nH_n(-1)$$

$$e^{2\overline{g}(-1)L} \qquad 4.1.16$$

Since the intensity profile is even, in this case it is assumed that the gain function $\overline{g}$ is even too, and the approximation then simplifies to

$$\lambda(h+\Sigma c_n H_n(x)) = he^{2\overline{g}/(y_0)L}+he^{2\overline{g}(1)L}H_1(x)$$

$$+\sum_{n=1}^{N} e^{2\overline{g}(y_0^a)} a_n F_{n+1}(x) + \sum_{n=1}^{N} e^{2\overline{g}(y_0^b)} a_n G_{n+1}(x)$$

$$+H_1(x)e^{2\overline{g}(1)L}\sum_{n=1}^{N} c_n H_n(1) \qquad 4.1.17$$

Equating coefficients of $H_n(x)$, $n \neq 1$ shows that

$$\lambda c_{n+1} = a_n e^{2\overline{g}(y_0^a)L} F_n(x)+b_n e^{2\overline{g}(y_0^b)L} G_n(x) \qquad 4.1.18$$

However, y_0^a and y_0^b themselves are now functions of x , and therefore, the sequential arguments leading up to an eigenvalue polynomial can no longer be made.

In order to build that polynomial, one more simplifying assumption is made, that being whatever intensity fluctuations present across the output plane exist, their effect on the gain is negligible. The gain factor is then assumed to be a constant, for all points across the resonator.

Defining the gain factor

$$\xi = e^{2\overline{g}L} \qquad 4.1.19$$

where $\bar{g}$ is given by 4.1.4, then, the equation becomes

$$\lambda\left(h+\sum_{n=1}^{N} c_n H_n(x)\right) = h\xi+h\xi H_1(x)+\sum_{n=1}^{N}\xi c_n H_{n+1}(x)$$

$$+H_1(x)\sum_{n=1}^{N}\xi c_n H_n(1) \qquad 4.1.20$$

Equating coefficients of $H_n(x)$ now gives

$$c_{n+1} = \frac{\xi}{\lambda} c_n \qquad 4.1.21$$

$$c_2 = \frac{\xi}{\lambda} c_1 \qquad 4.1.22$$

$$c_{n+1} = c_1\left(\frac{\xi}{\lambda}\right)^n \qquad 4.1.22$$

and in turn it is seen that

$$c_n\left(\frac{\lambda}{\xi}\right)^n = c_1\frac{\lambda}{\xi} = c_N\left(\frac{\lambda}{\xi}\right)^N \qquad 4.1.23$$

Equating coefficients of $H\ (x)$ yields as before

$$\lambda c_1 = h\xi+\sum_{n=1}^{N}\xi c_n H_n(1) \qquad 4.1.24$$

Equating constant terms shows that

$$\lambda h = h\xi+\xi c_N H_{N+1} \qquad 4.1.25$$

$$\frac{\lambda h - h\xi}{\xi H_{N+1}} = c_n \qquad 4.1.26$$

$$\frac{h(\lambda-\xi)}{\xi H_{N+1}} = c_n \qquad 4.1.27$$

Therefore, from 4.1.27 and 4.1.23,

$$c_n = \left(\frac{\lambda}{\xi}\right)^{N-n} \frac{(\lambda-\xi)h}{H_{N+1}} \qquad 4.1.28$$

Substituting this into 4.1.24 yields

$$\lambda\left(\frac{\lambda}{\xi}\right)^{N-1} \frac{h(\lambda-\xi)}{\xi H_{N+1}} = h\xi + \sum_{n=1}^{N} \xi\left(\frac{\lambda}{\xi}\right)^{N-n} \frac{h(\lambda-\xi)}{\xi H_{N+1}} H_n(1) \qquad 4.1.29$$

which simplifies to

$$\left(\frac{\lambda}{\xi}\right)^{N}(\lambda-\xi) = \xi H_{N+1} + (\lambda-\xi)\sum_{n=1}^{N} \left(\frac{\lambda}{\xi}\right)^{N-n} H_n(1) \qquad 4.1.30$$

and the polynomial is then given by

$$\lambda^{N}(\lambda-\xi) = \xi^{N+1} H_{N+1} + (\lambda-\xi)\sum_{n=1}^{N} \lambda^{N-n}\xi^{N}\xi^{n-N} H_n(1) \qquad 4.1.31$$

$$= \xi^{N+1} H_{N+1} + (\lambda-\xi) \sum_{n=1}^{N} \lambda^{N-n}\xi^{n} H_n(1) \qquad 4.1.32$$

The roots of this polynomial can be found using the same method as before, which will then be rough approximations

to mode eigenvalues. If the model is to be correct for a steady state resonator, gain should just balance loss in that resonator, implying that

$$u'(x) = u(x) \qquad 4.1.33$$

and therefore

$$\gamma = 1 \qquad 4.1.34$$

If $\gamma=1$, then from 2.1.32

$$\lambda = \sqrt{m} \qquad 4.1.35$$

Using this condition, 4.1.35, then h can be modified until the lowest loss mode has an eigenvalue equal to $\sqrt{m}$. The value of h that allows this should then be the most reasonable value of h .

When h is found, the proper gain factor is in turn found by substituting h^2 for the intensity ratio in 4.1.4.

To extend these solutions beyond the shadow boundaries, one merely has to multiply the constants a_n and b_n in the expressions derived in the last section of the previous chapter by ξ , and change the constant factor to h .

V. Results

Implementation of Code and Result Check

The expressions developed in chapters three and four were incorporated into a CDC Fortran IV program, BARC, which was organized into two basic sections. The first section included development of the coefficients of the eigenvalue polynomials 3.2.15 and 3.2.26, the computation of the roots through IMSL routine ZCPOLY, the computation of the weighting constants c_n according to 3.2.16 and 3.1.23, and the preliminary eigenvalue expressions based on 3.1.2. The second part, in a separate subroutine, implemented the expressions developed in the third section of chapter three: the eigenvalue expressions valid for all x . The program was then run for various cavity parameters and the results were compared with the results of other programs (Ref 6:1536; Ref 8:239).

Table 1 represents a comparison of eigenvalue moduli resulting from the program developed in this work, and those from the Moore and McCarthy program.

These results are for a cavity with magnification of 2.9 and an effective Fresnel number of 16.4 . The solution compared is that of the even parity case.

It is seen that the two codes predict modes with very

TABLE 1

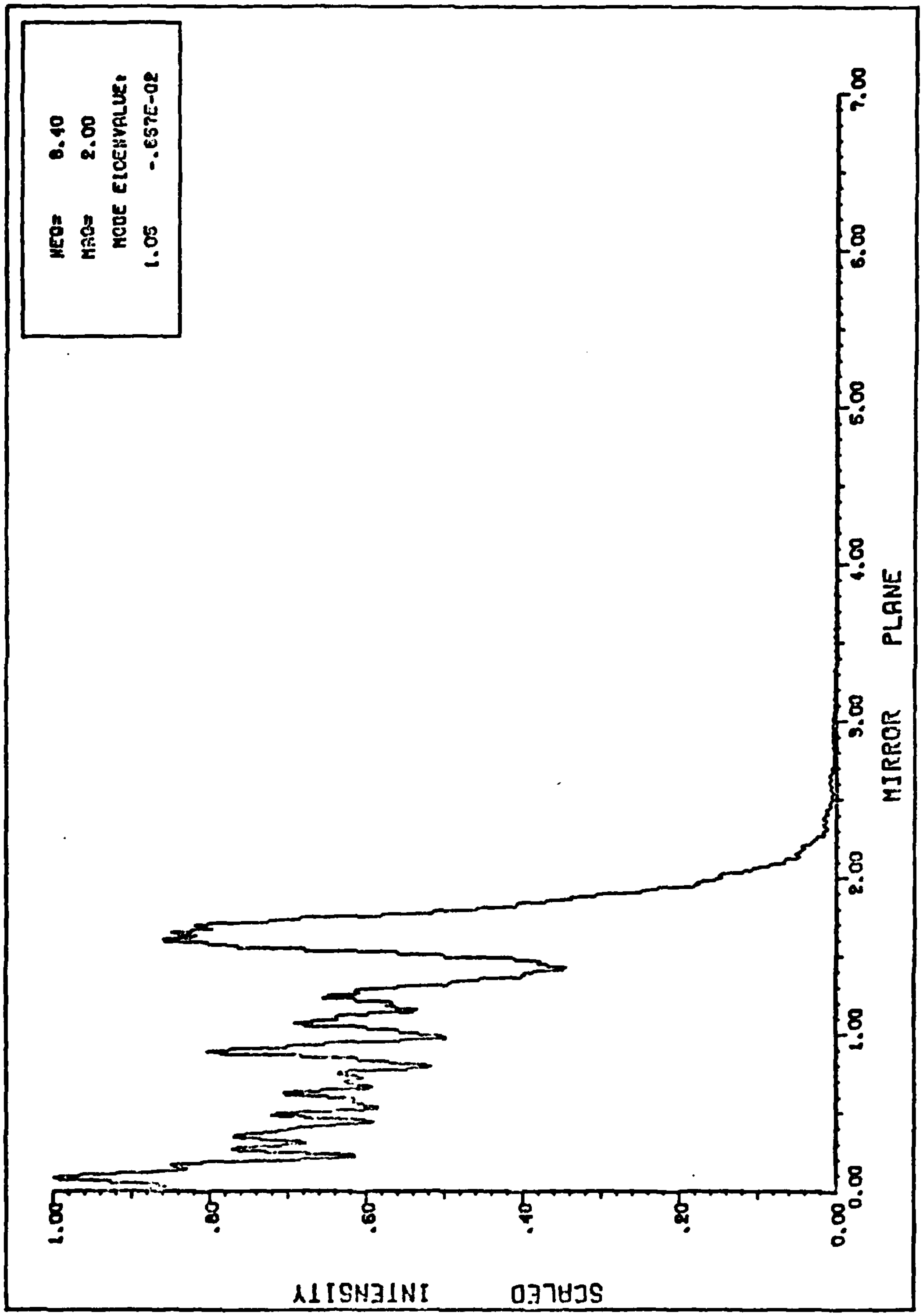
Mode	Mod BARC	Mod m&m[c]
1	1.040105	1.040105
2	..625501	.625501
3	.606668	.606668
4	.496561	.496561
5	.467285	.467285
6	.157664	.139999

similar losses, since

$$\text{loss} = 1 - \frac{\lambda^*\lambda}{m^2} \qquad 5.1.1$$

and the $\lambda^*\lambda$ values are all very close.

Figures 3 through 8 are included to show results of eigenfunction intensity plots over similar ranges for the Moore and McCarthy program and program BARC. Figures 10 through 15 show comparison between BARC's results and those published in reference 6 (Ref 6:1536-1539). In both cases, through visual comparison, program BARC produces results that are very similar to results from previous methods. This indicates that BARC produces valid results, at least to the extent that the previous methods are valid.



EVEN PARITY MODE # 1 APPROX # 2

Figure 3

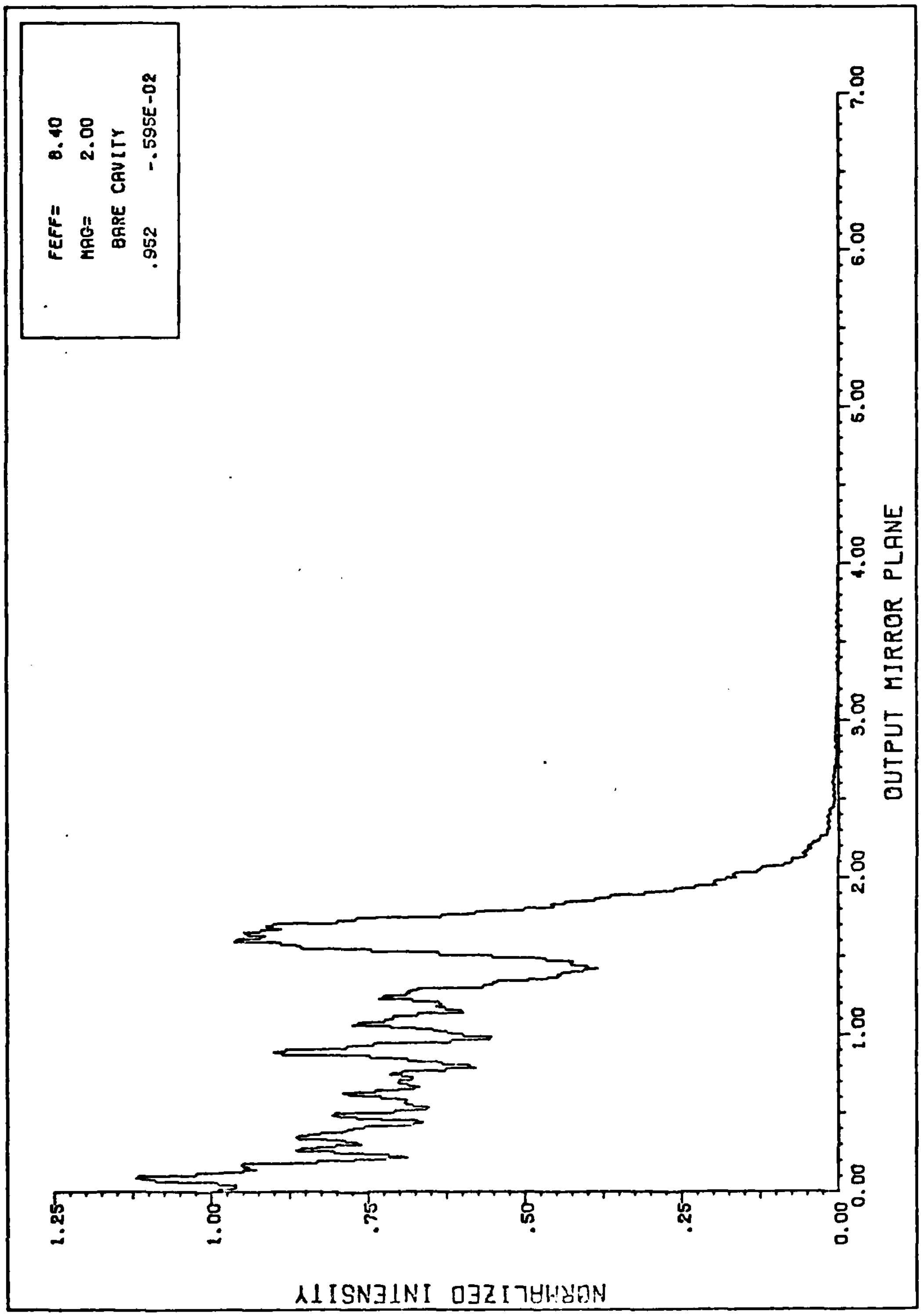
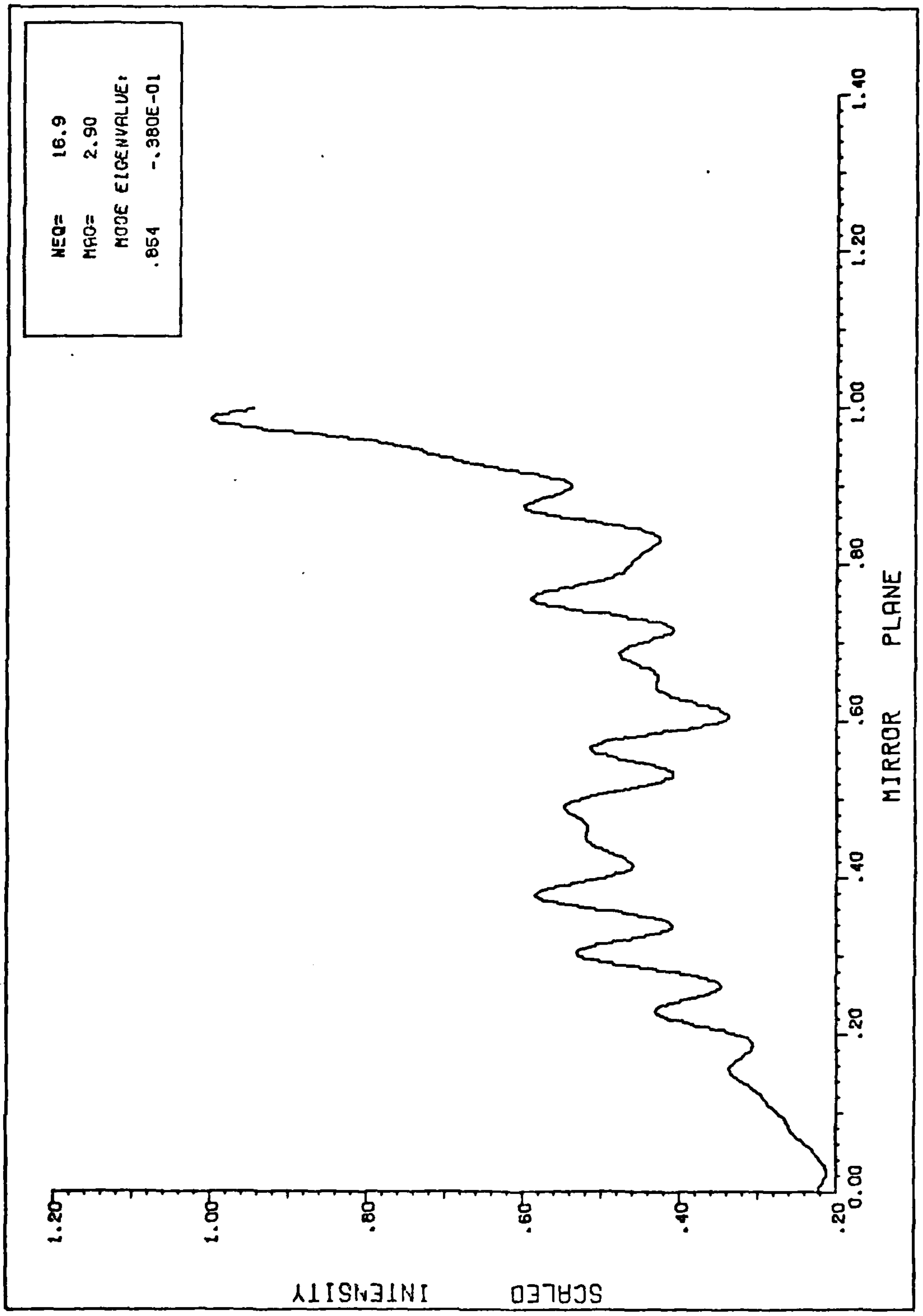
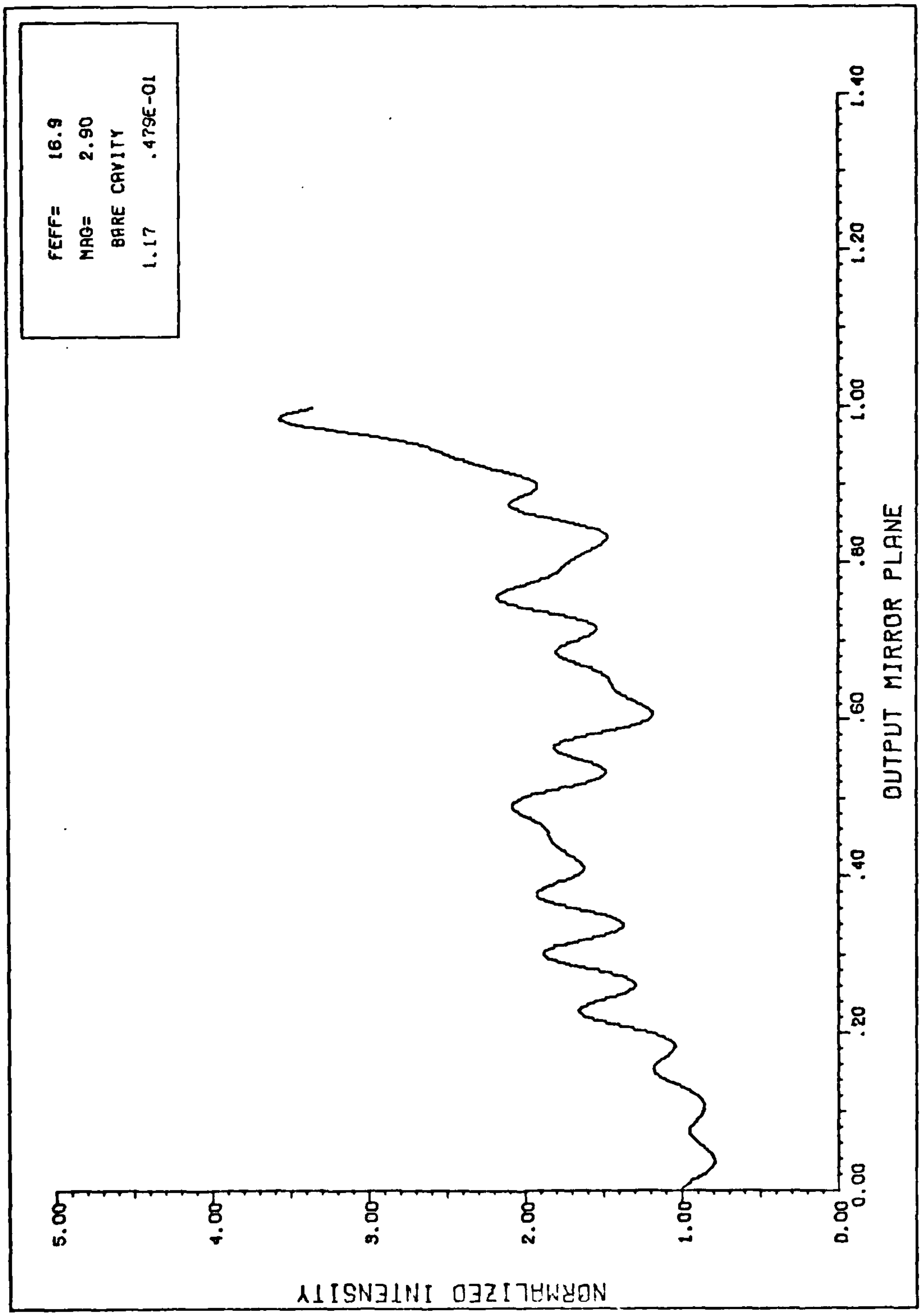

FEFF= 8.40
MAG= 2.00
BARE CAVITY
.952 -.595E-02
OUTPUT MIRROR PLANE
NORMALIZED INTENSITY
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
.25
.50
.75
1.00
1.25

Figure 4



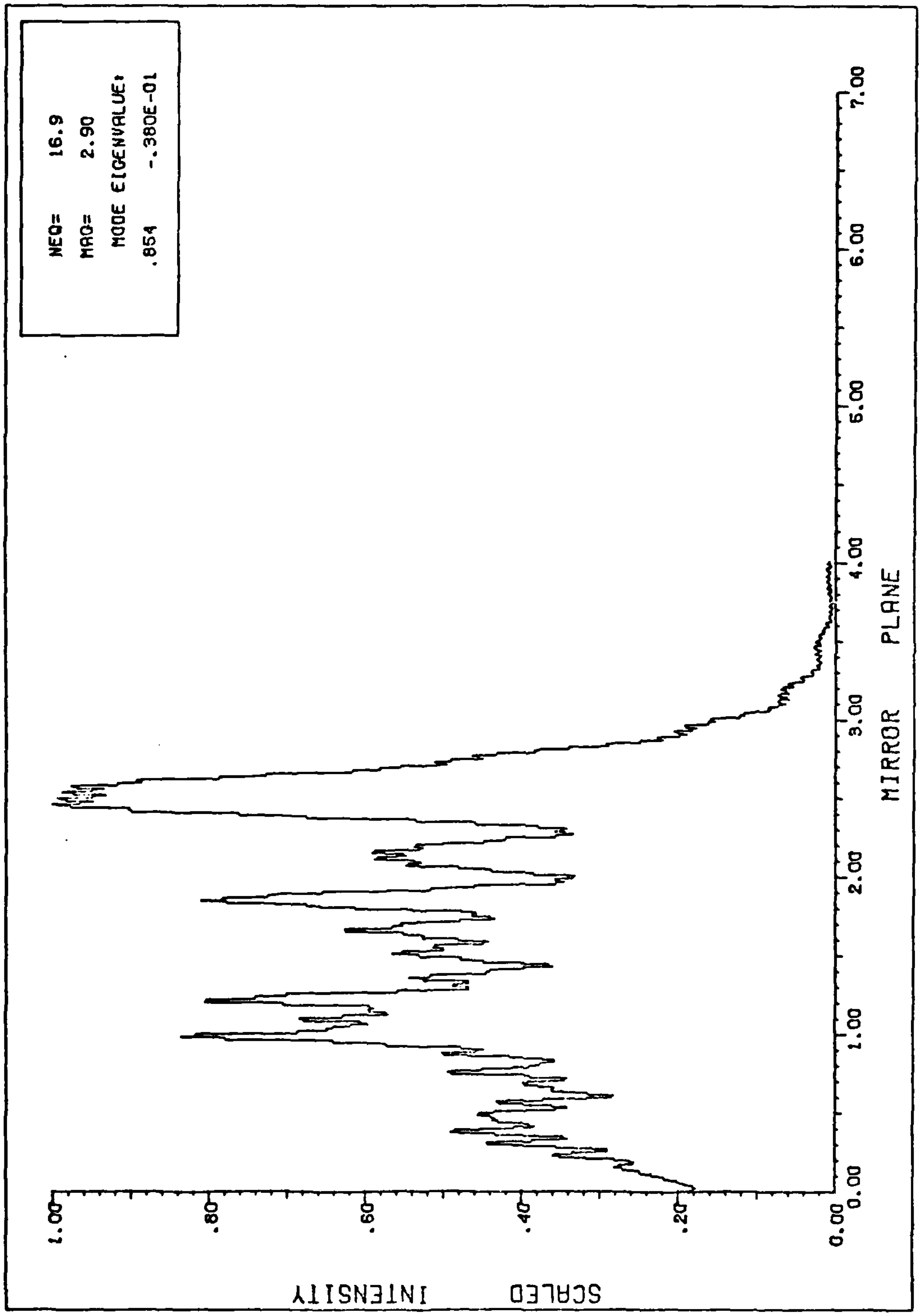
EVEN PARITY MODE # 1 APPROX # 1

Figure 5



FEFF= 16.9
MAG= 2.90
BARE CAVITY
1.17 .479E-01
NORMALIZED INTENSITY
0.00
1.00
2.00
3.00
4.00
5.00
OUTPUT MIRROR PLANE
0.00
.20
.40
.60
.80
1.00
1.20
1.40

Figure 6



EVEN PARITY MODE # 1 APPROX # 2

Figure 7

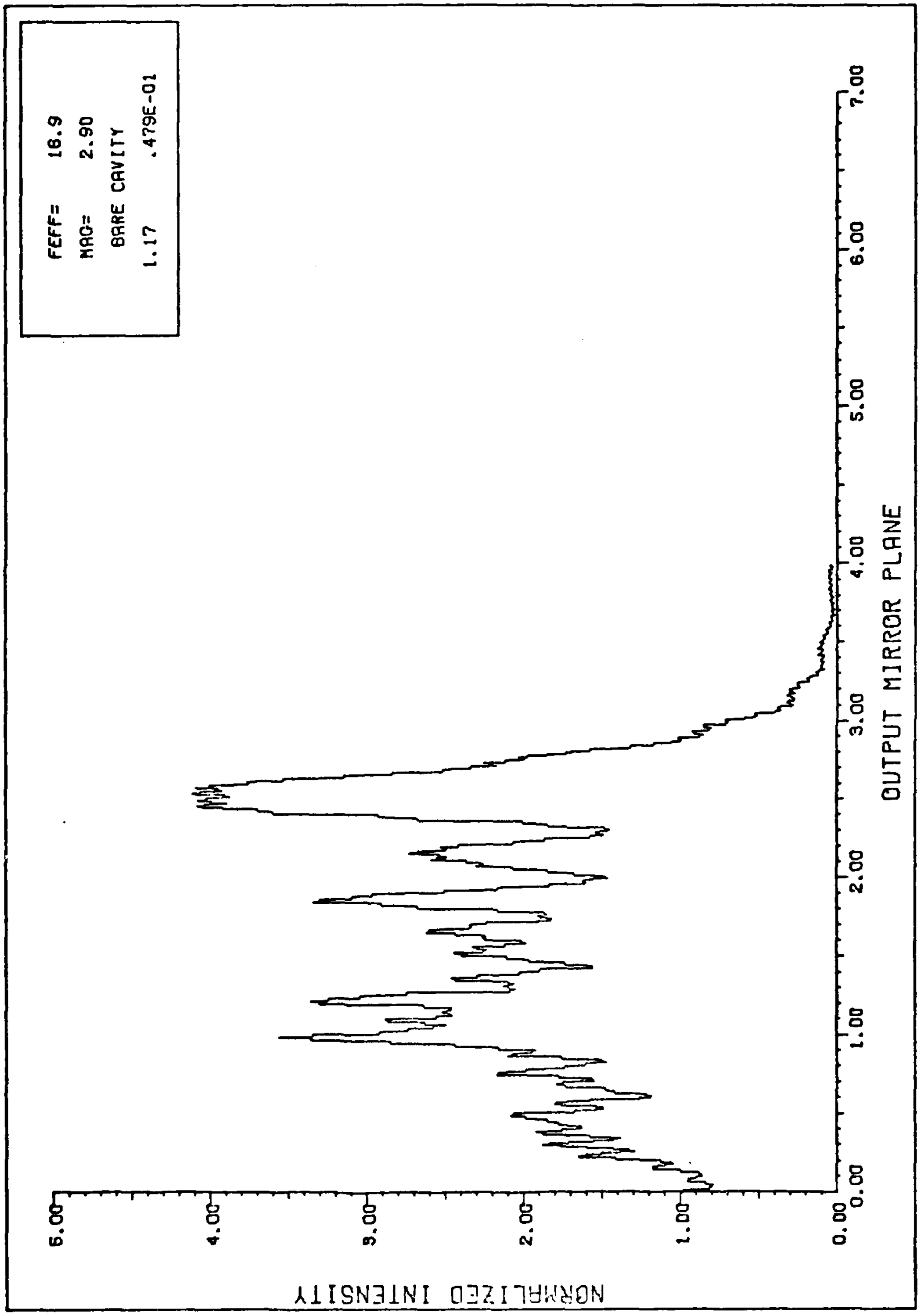
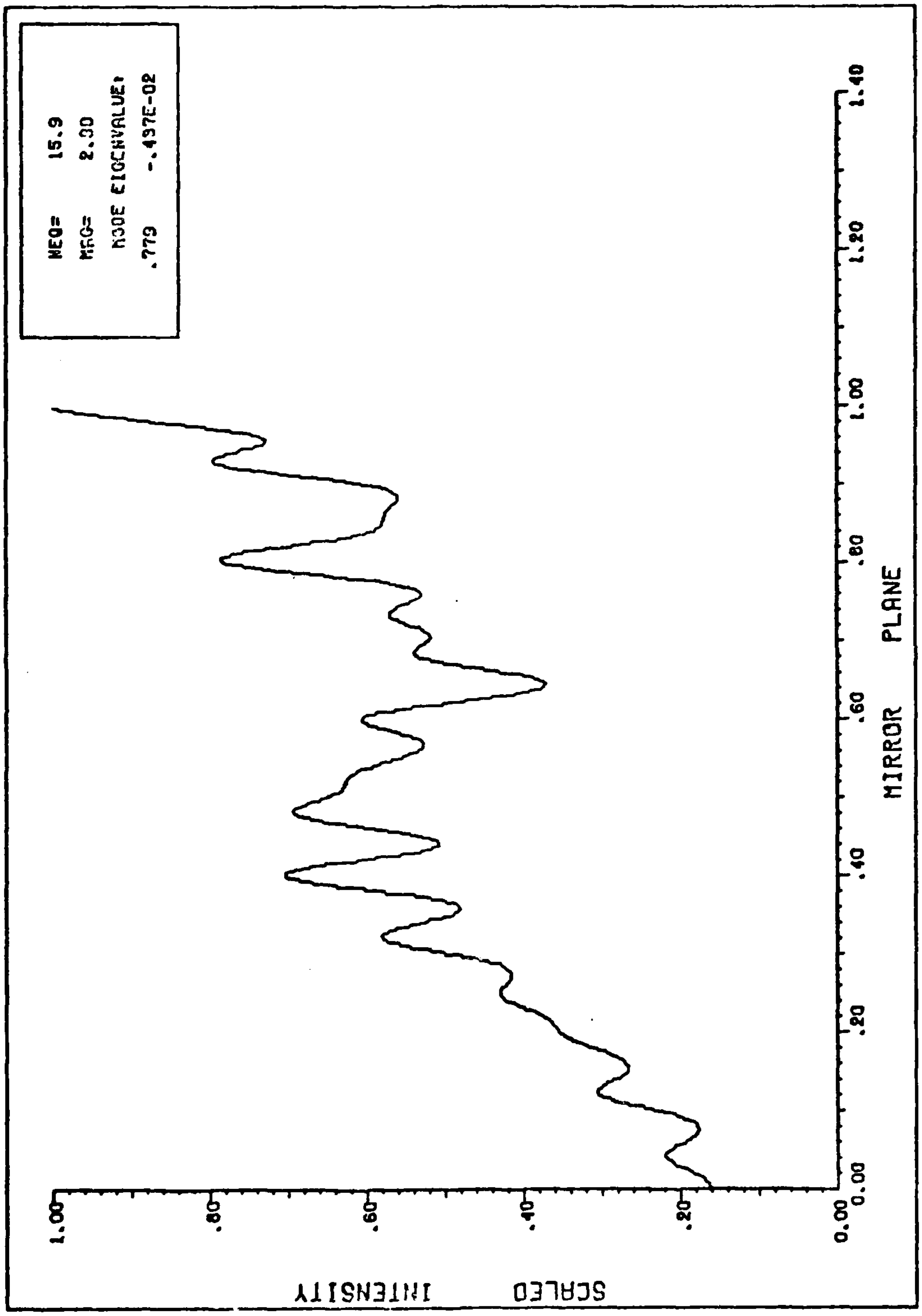
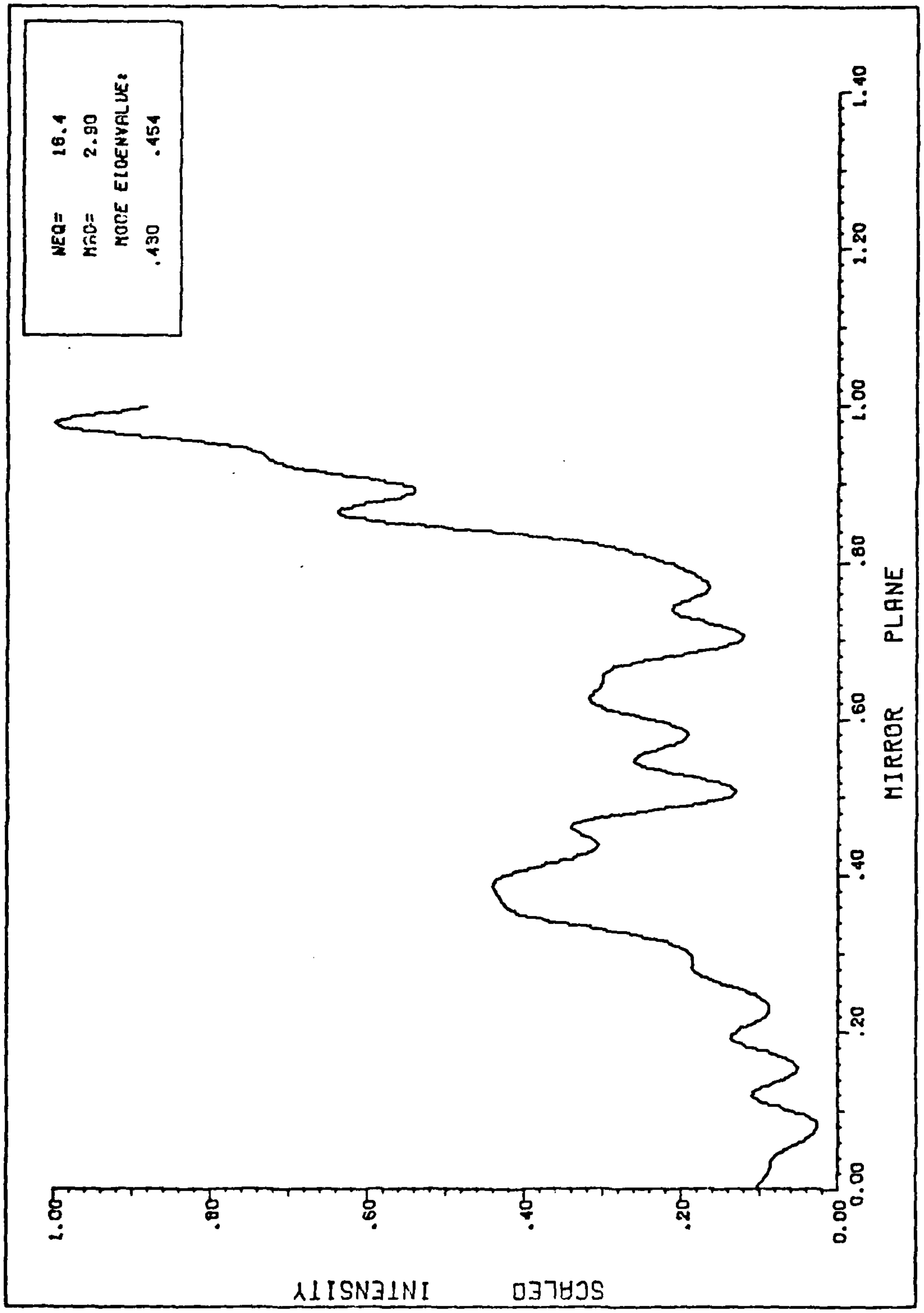

FEFF= 18.9
MAG= 2.90
BARE CAVITY
1.17 .479E-01
OUTPUT MIRROR PLANE
NORMALIZED INTENSITY
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00

Figure 8



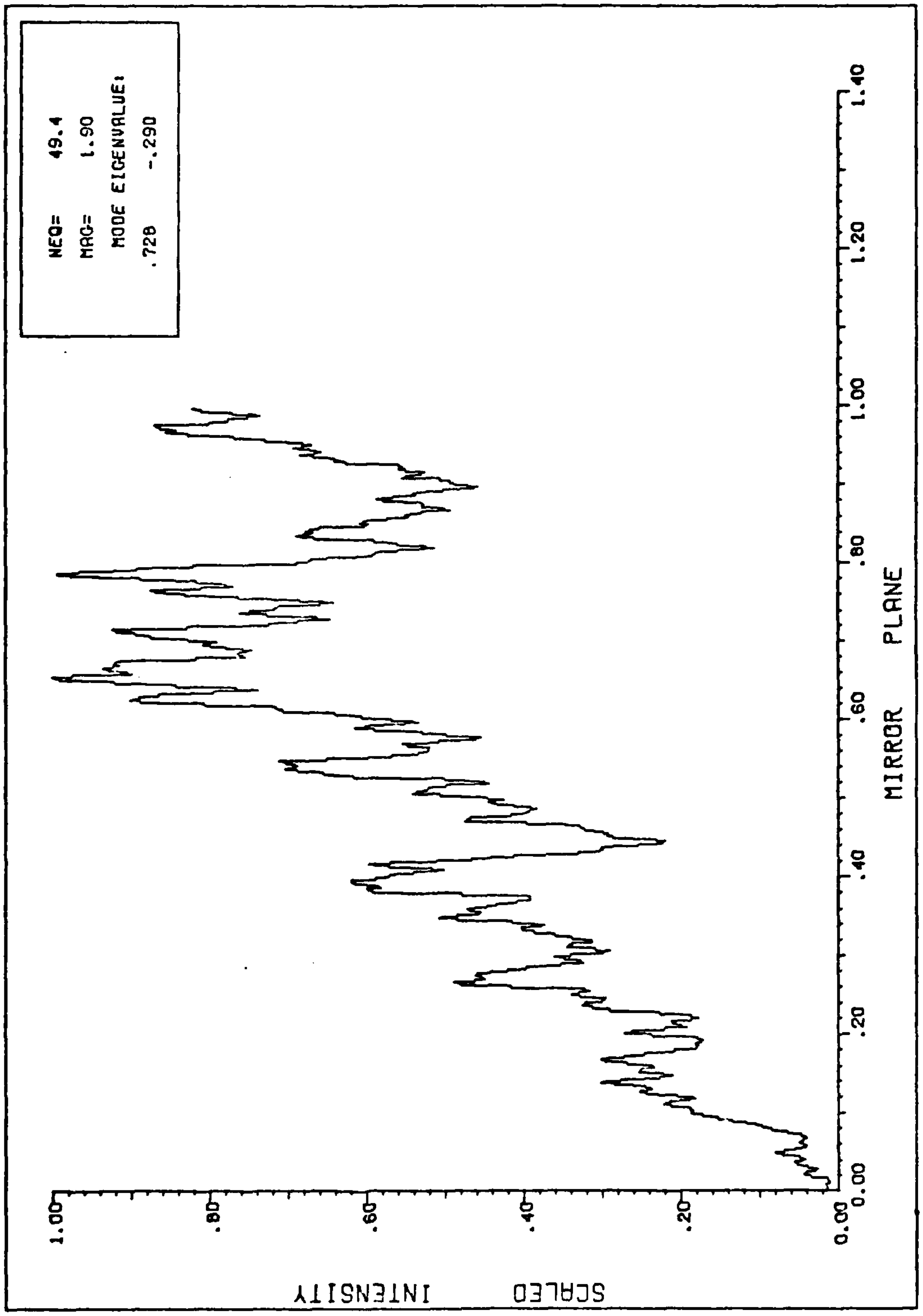
EVEN PARITY MODE # 2 APPROX # 1

Figure 9



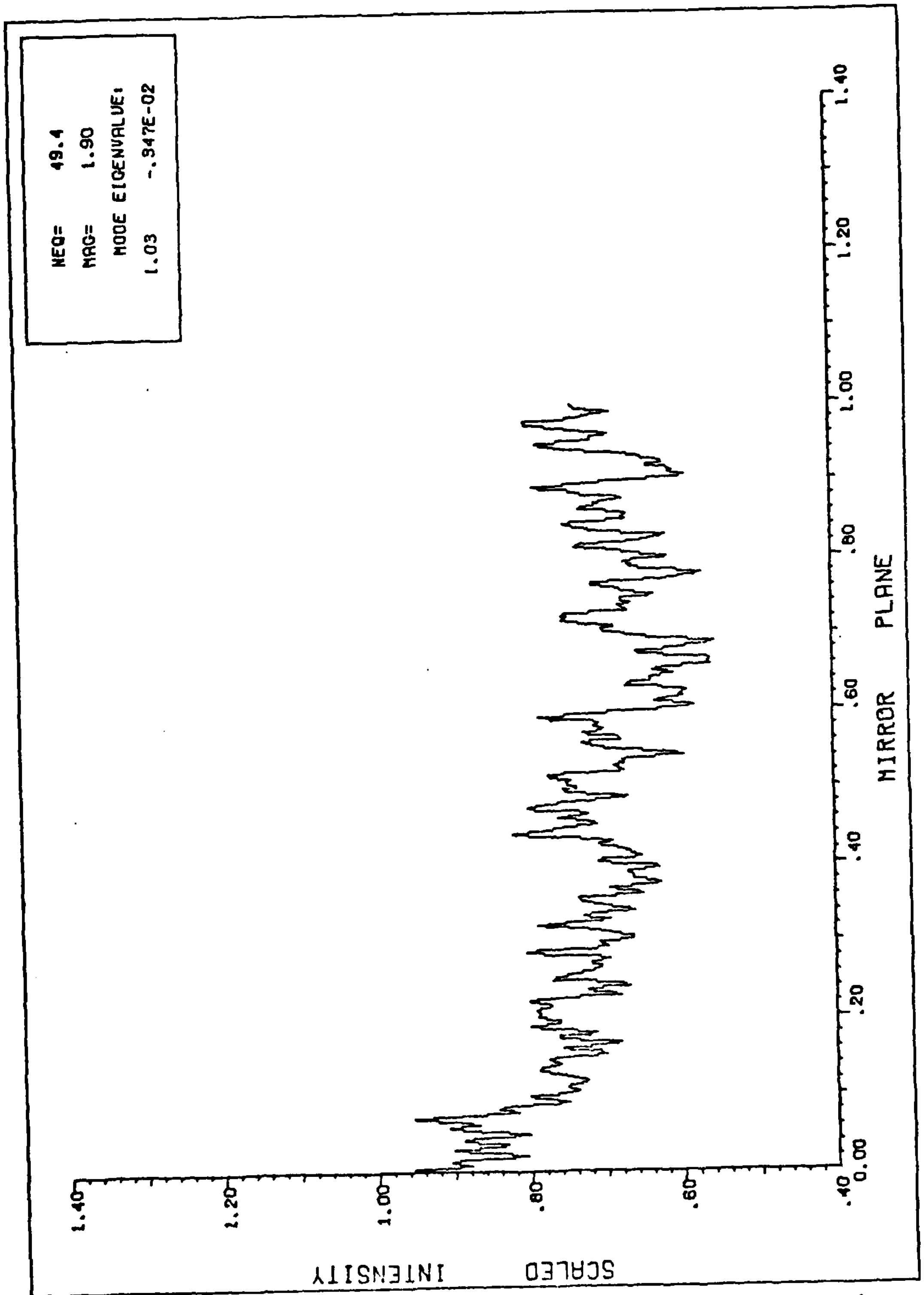
EVEN PARITY MODE # 2 APPROX # 1

Figure 10



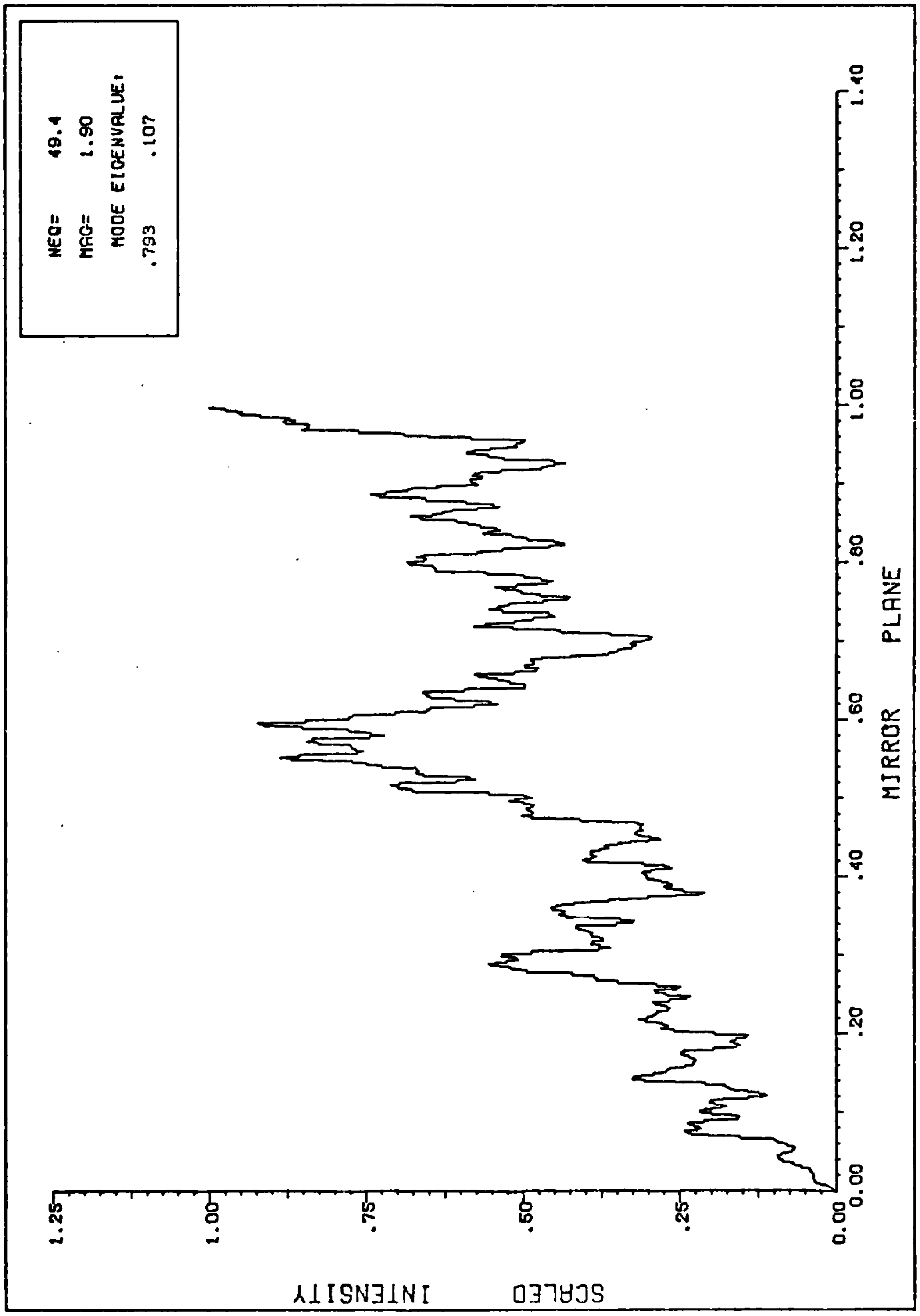
EVEN PARITY MODE # 2 APPROX # 1

Figure 11



EVEN PARITY MODE # 1 APPROX # 1

Figure 12



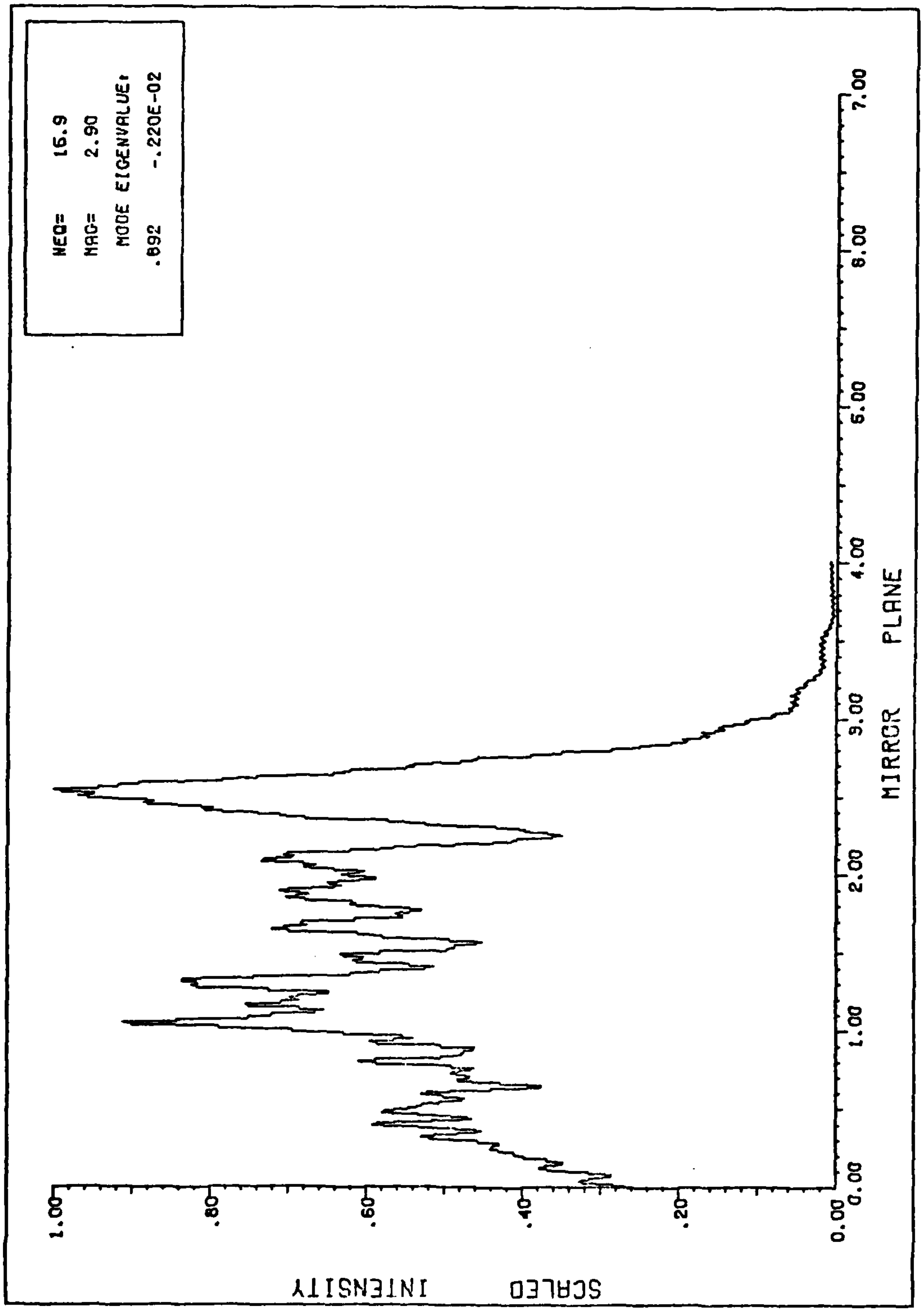
ODD PARITY MODE # 1 APPROX # 1

Figure 13

Bare Cavity and Gain Results

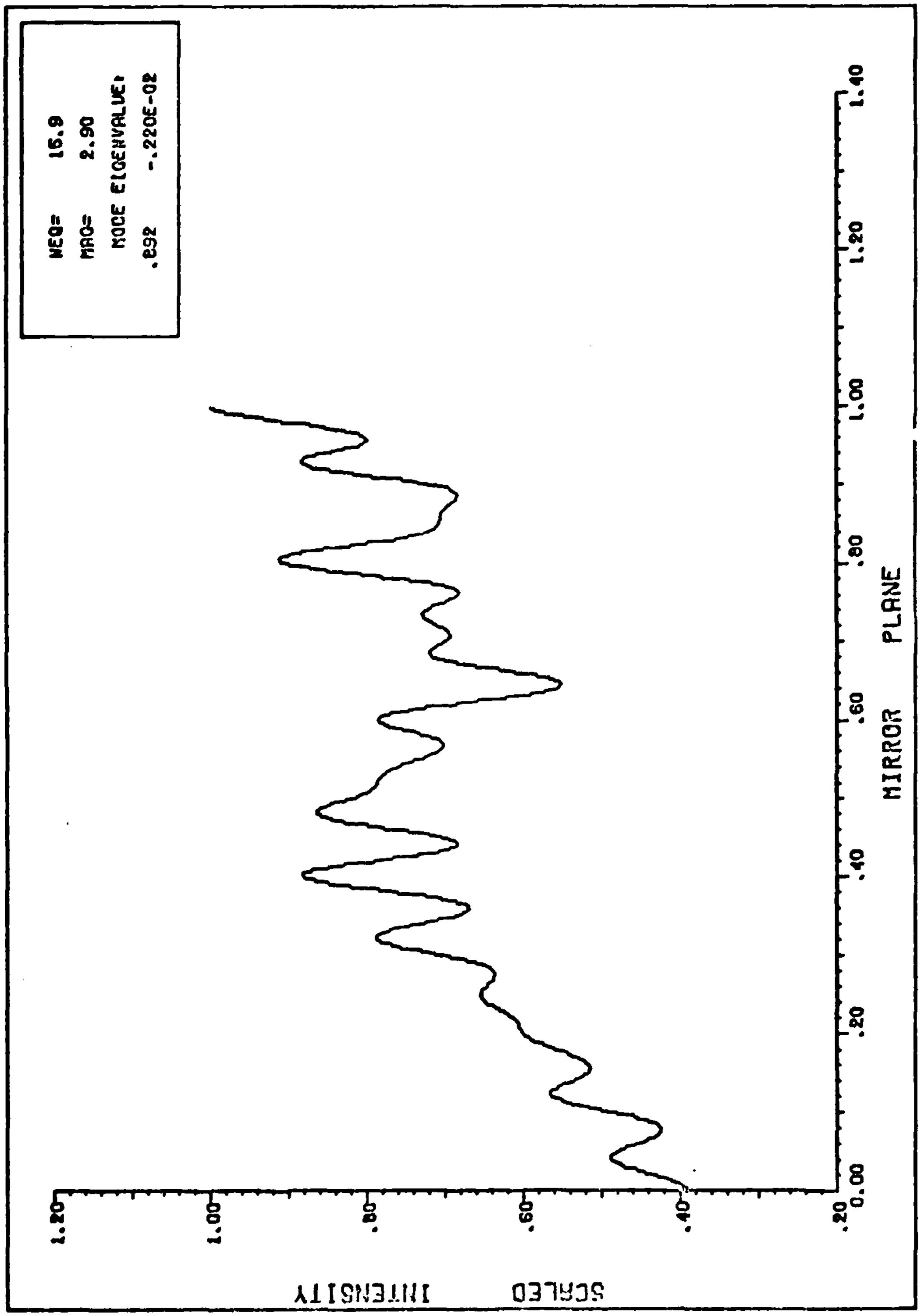
After the validity of the code was ascertained, the code was modified according to the expressions generated in chapter four. Figures 14 through 21 illustrate results obtained for a bare resonator of magnification 2.9 and equivalent fresnel numbers of 15.863 and 16.4 . These parameters are chosen to facilitate mode separation comparisons later in this section. Figures 22 through 29 illustrate results of a loaded cavity of the same configurational parameters but containing a gain medium of small signal gain $5\%cm^{-1}$ and cavity length of 200cm. This group of plots allows comparison between bare and loaded cavity cases. It is seen that this particular resonator model predicts that loaded cavity modes have nearly the same intensity profiles as bare cavity modes, differing only by a scale factor.

At first glance this seems reasonable, since in the bare cavity case, the whole eigenfunction was based on a wave of unit relative amplitude, and slight modifications on that wave by diffraction supplied by the oscillatory functions $H_n(x)$. In the loaded cavity, the eigenfunction is also based on a wave modified by the same functions, only the relative amplitude of that wave is no longer unity. Thus it seems likely that the profile would look moderately similar in both cases.



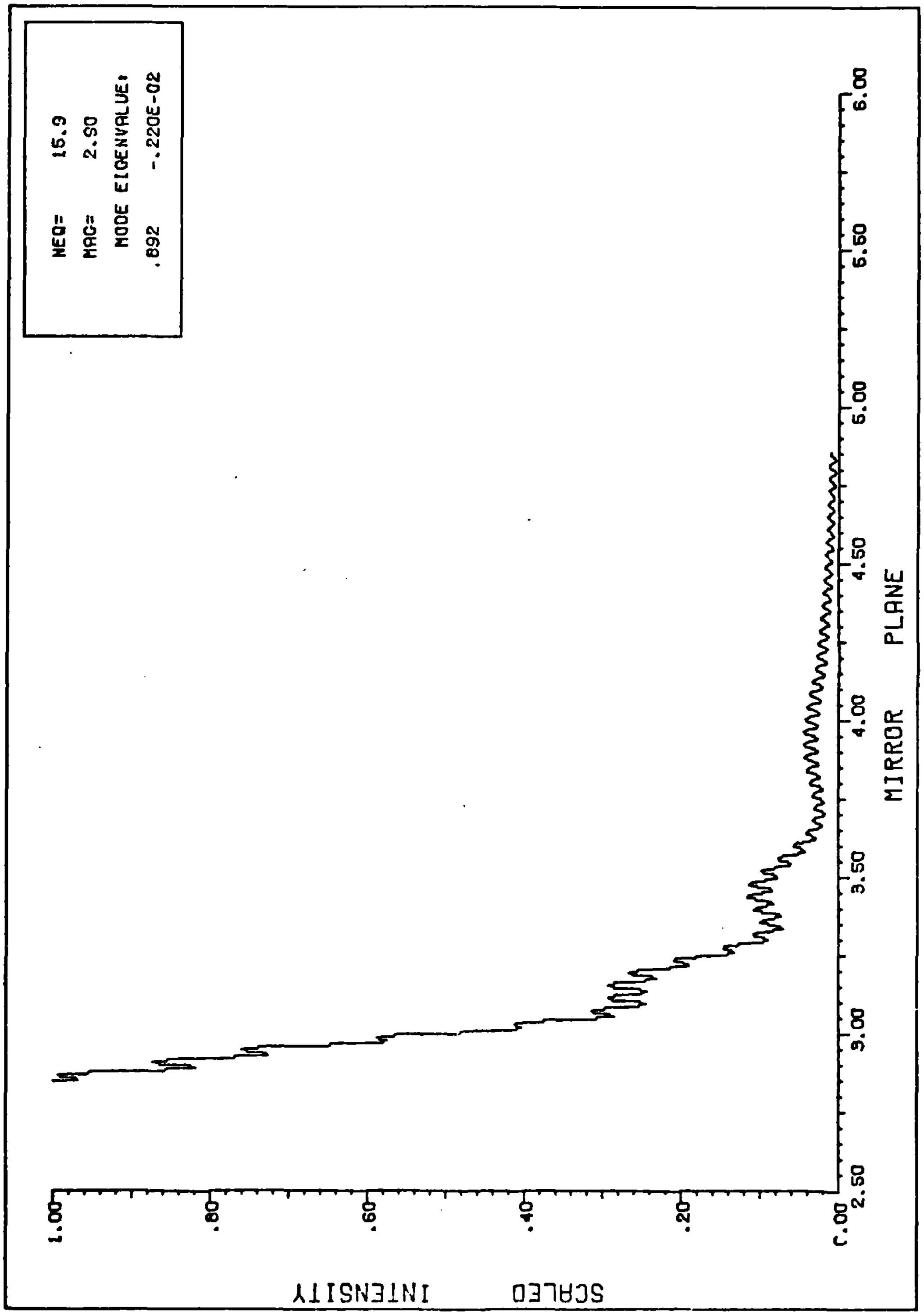
EVEN PARITY MODE # 1 APPROX # 2

Figure 14



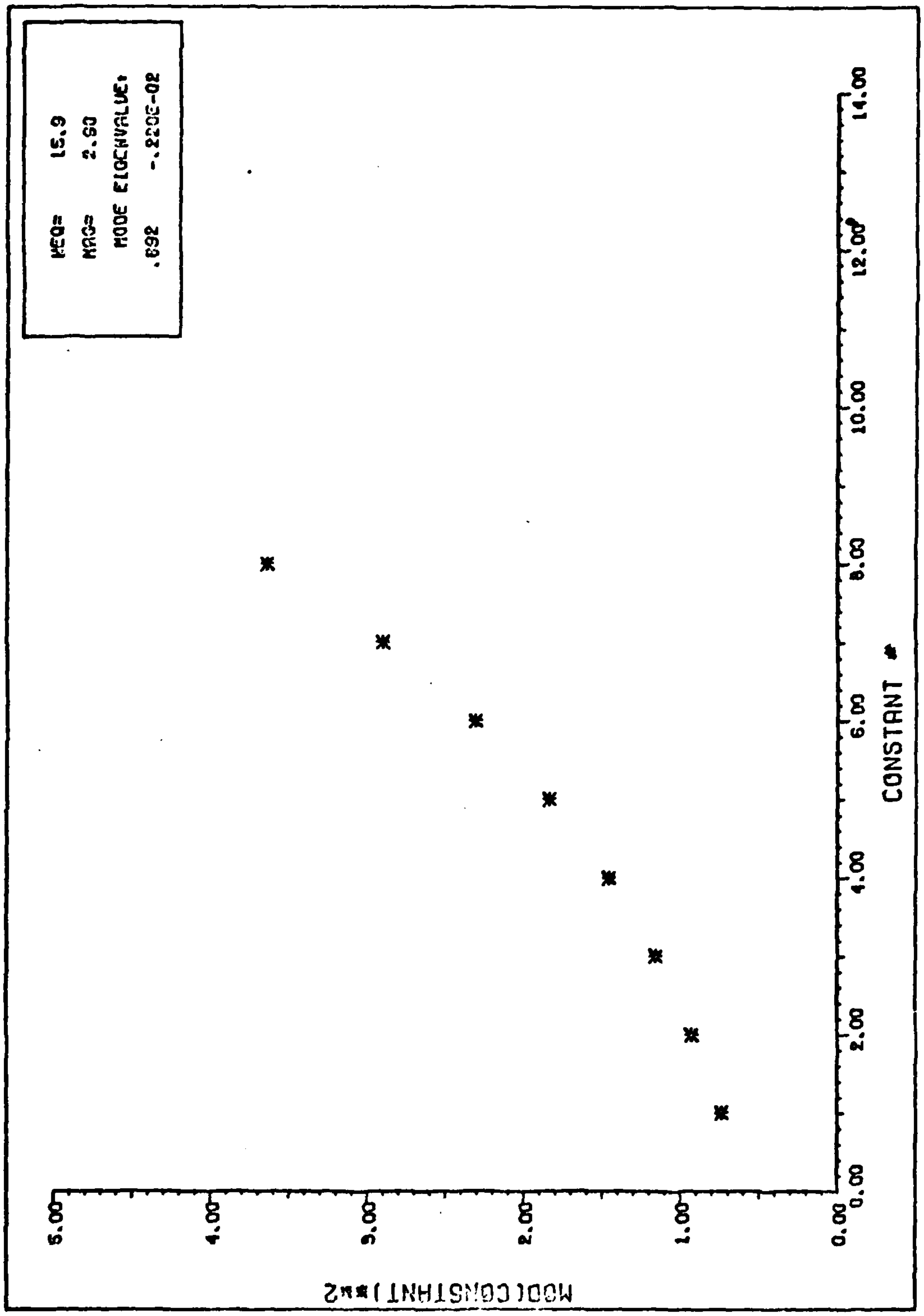
EVEN PARITY MODE # 1 APPROX # 1

Figure 15



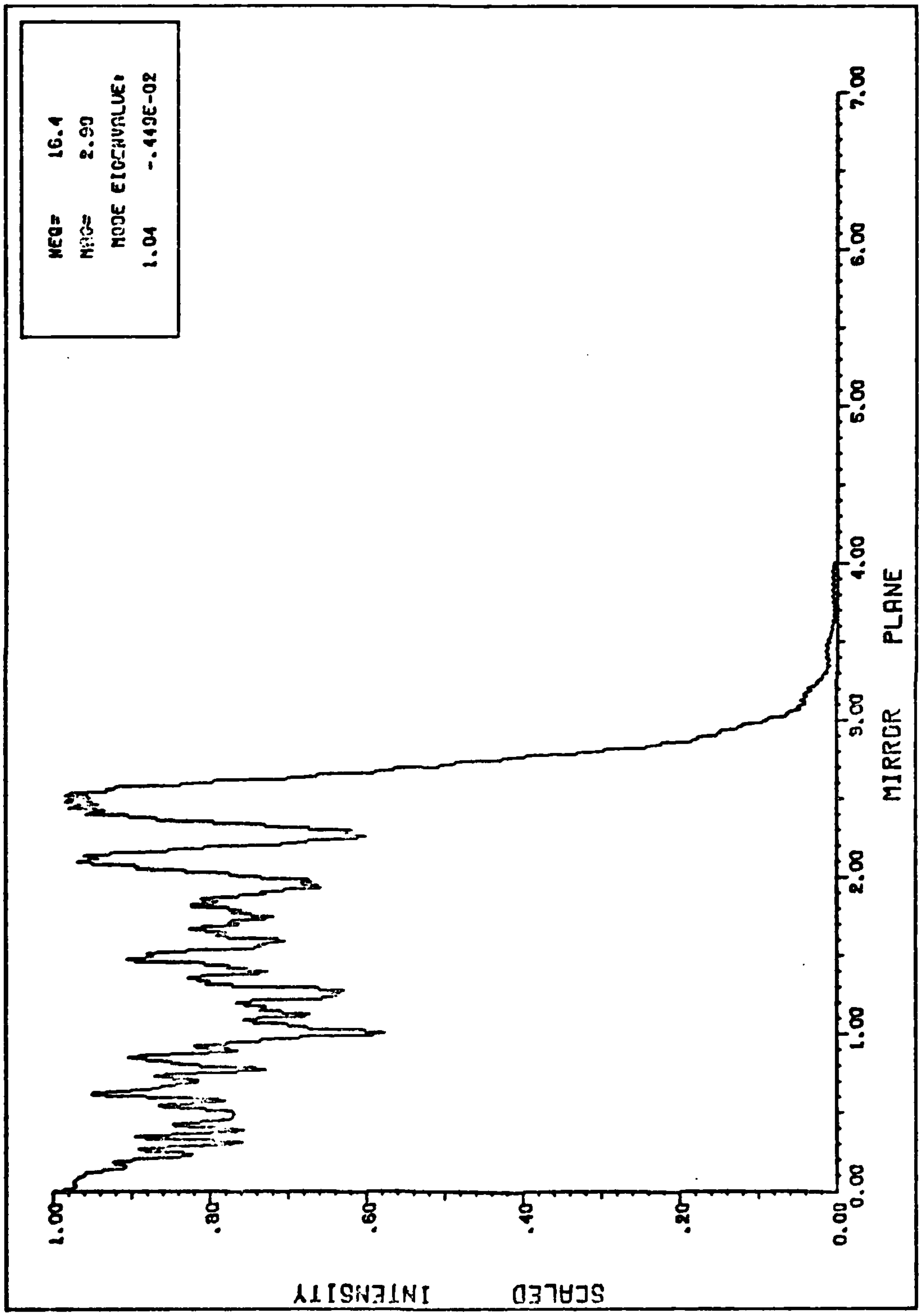
EVEN PARITY MODE # 1 APPROX # 2

Figure 16



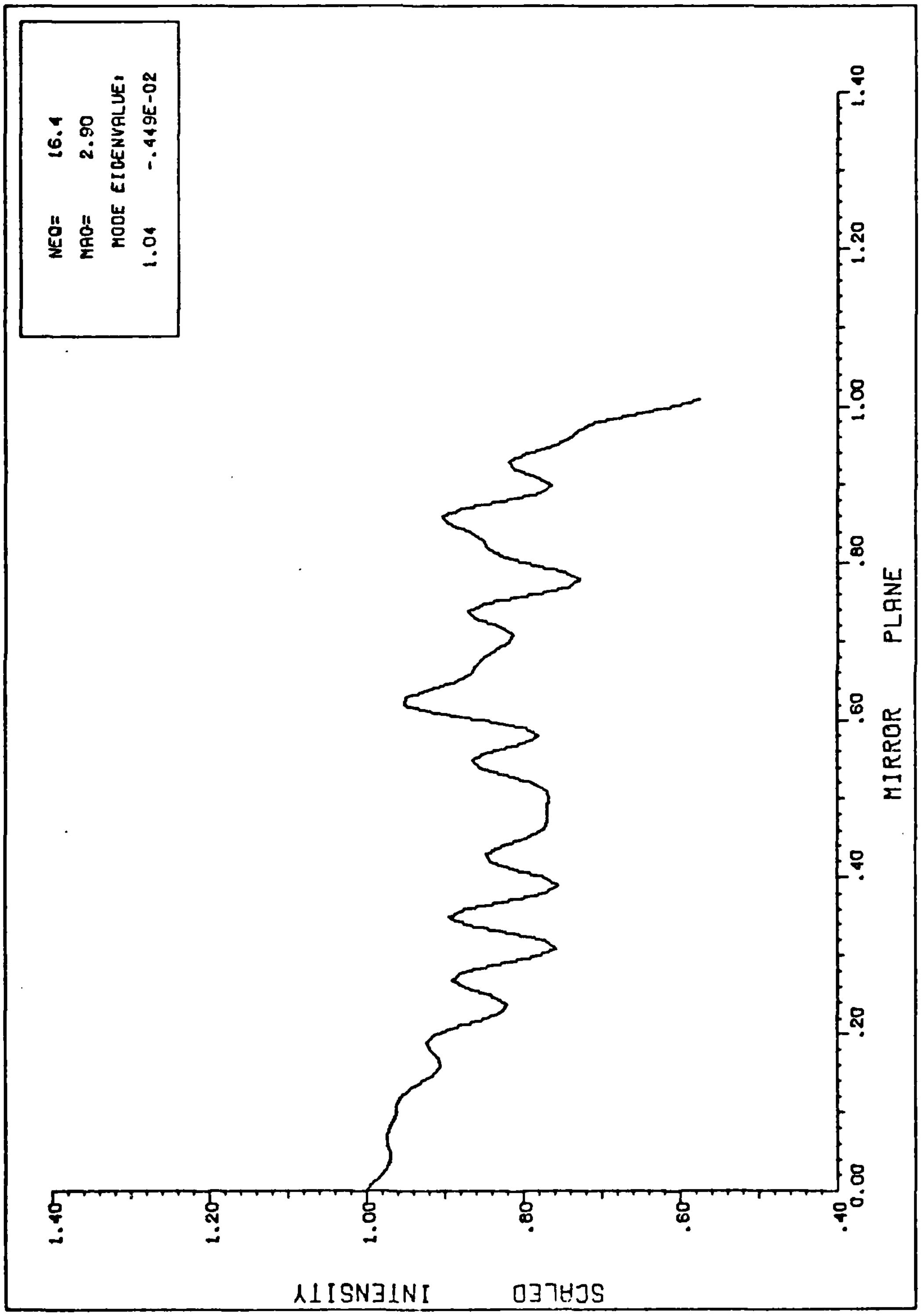
EVEN PARITY MODE # 1 APPROX # 1

Figure 17



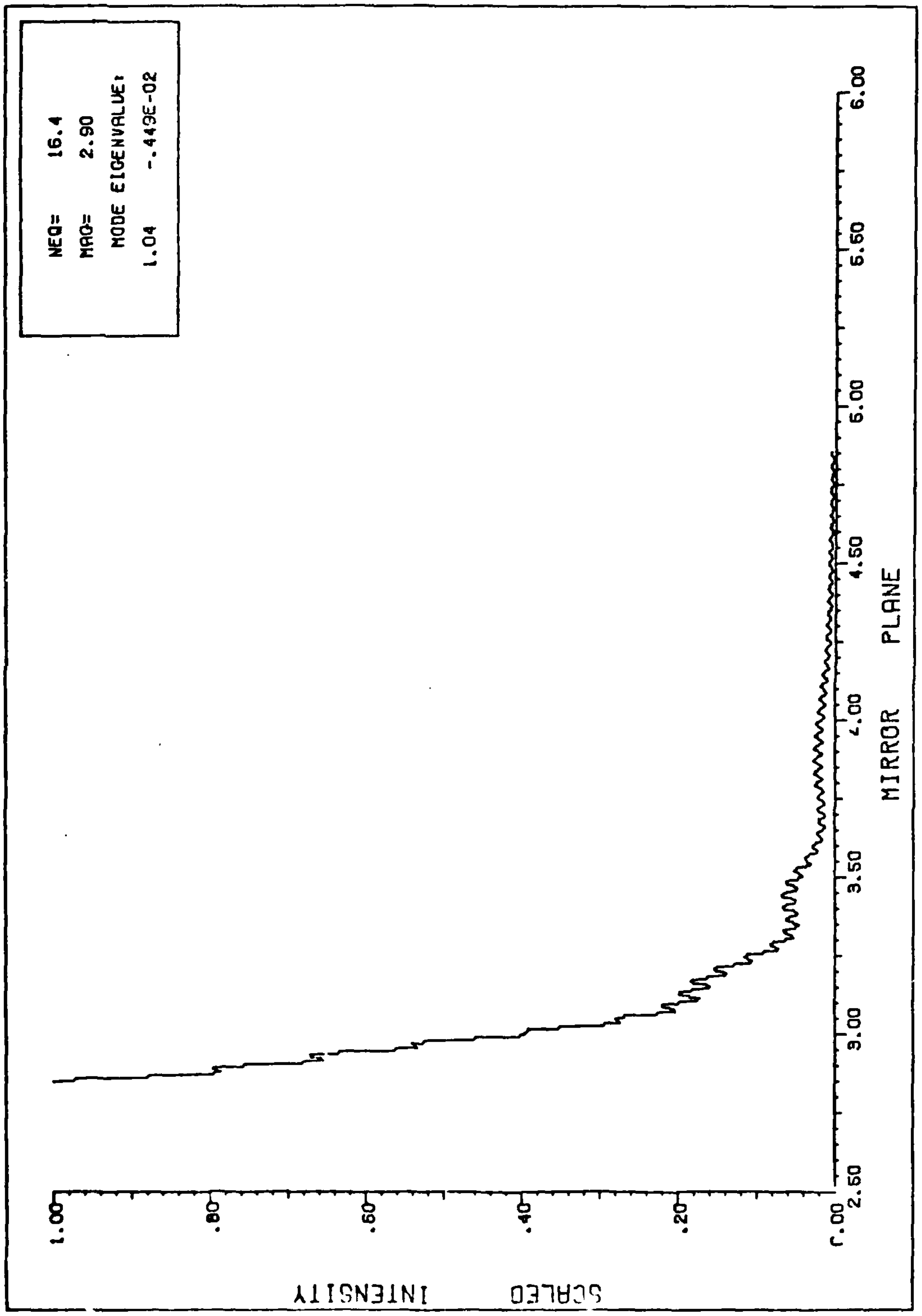
EVEN PARITY MODE # 1 APPROX # 2

Figure 18



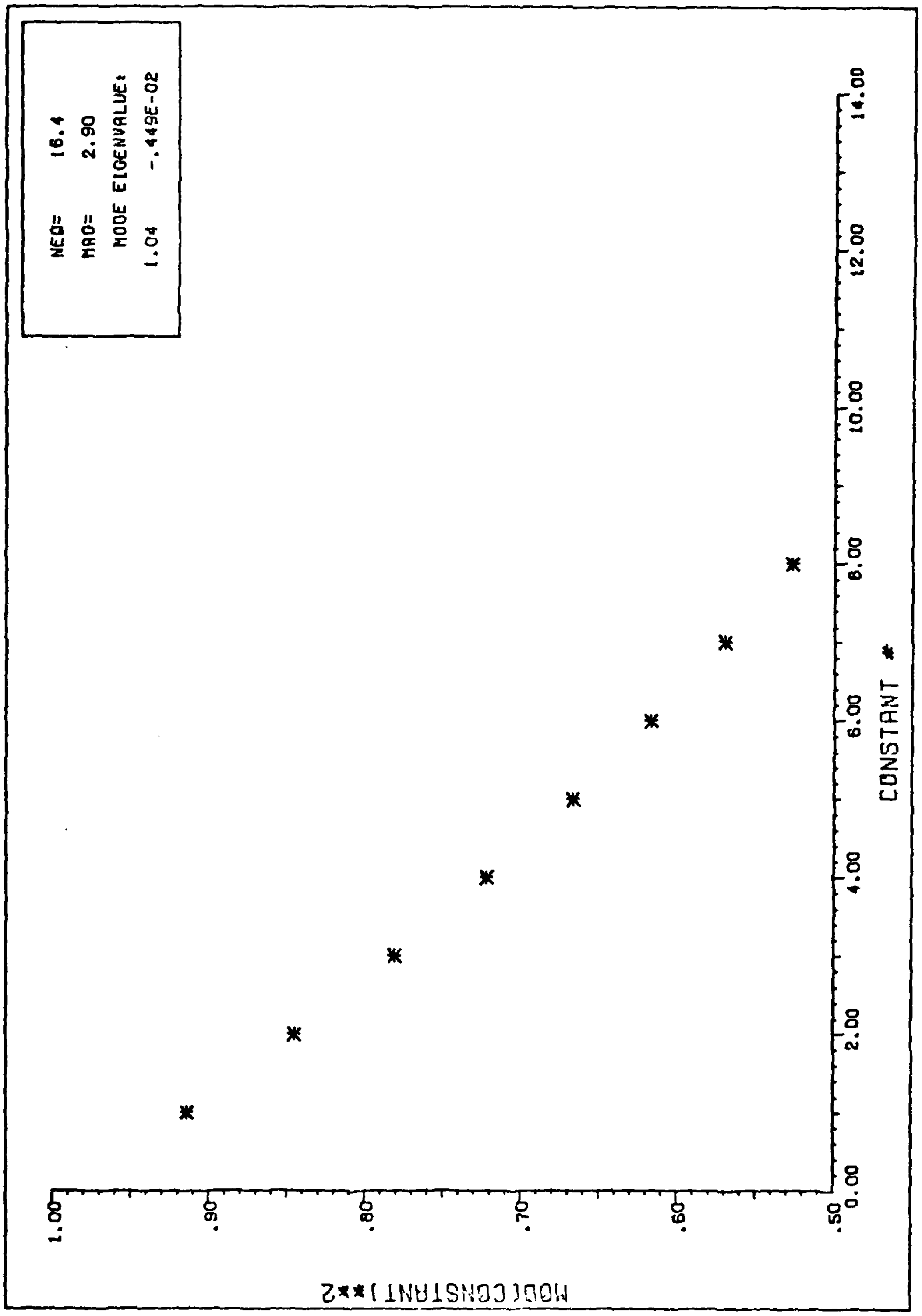
EVEN PARITY MODE # 1 APPROX # 2

Figure 19



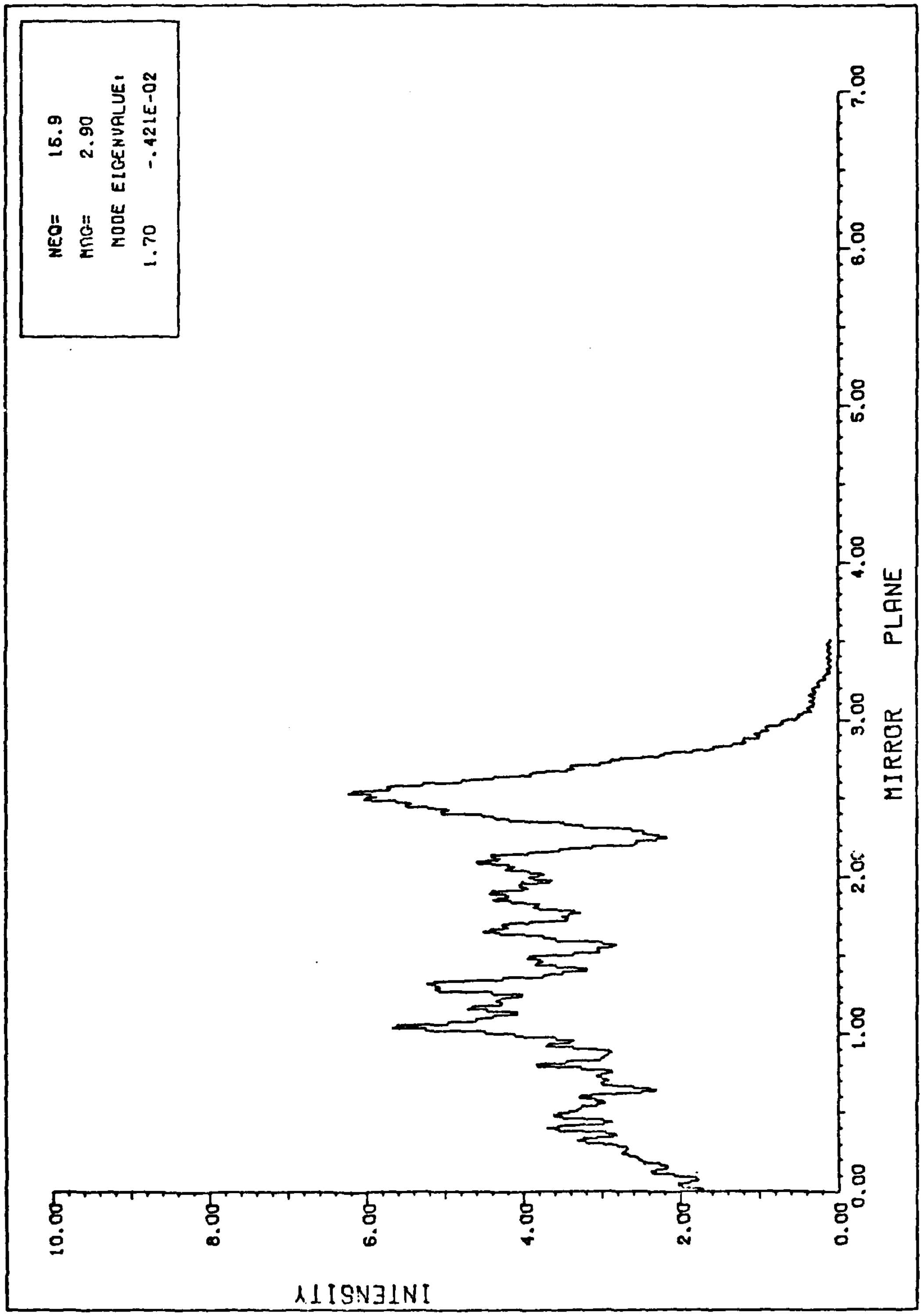
EVEN PARITY MODE # 1 APPROX # 2

Figure 20



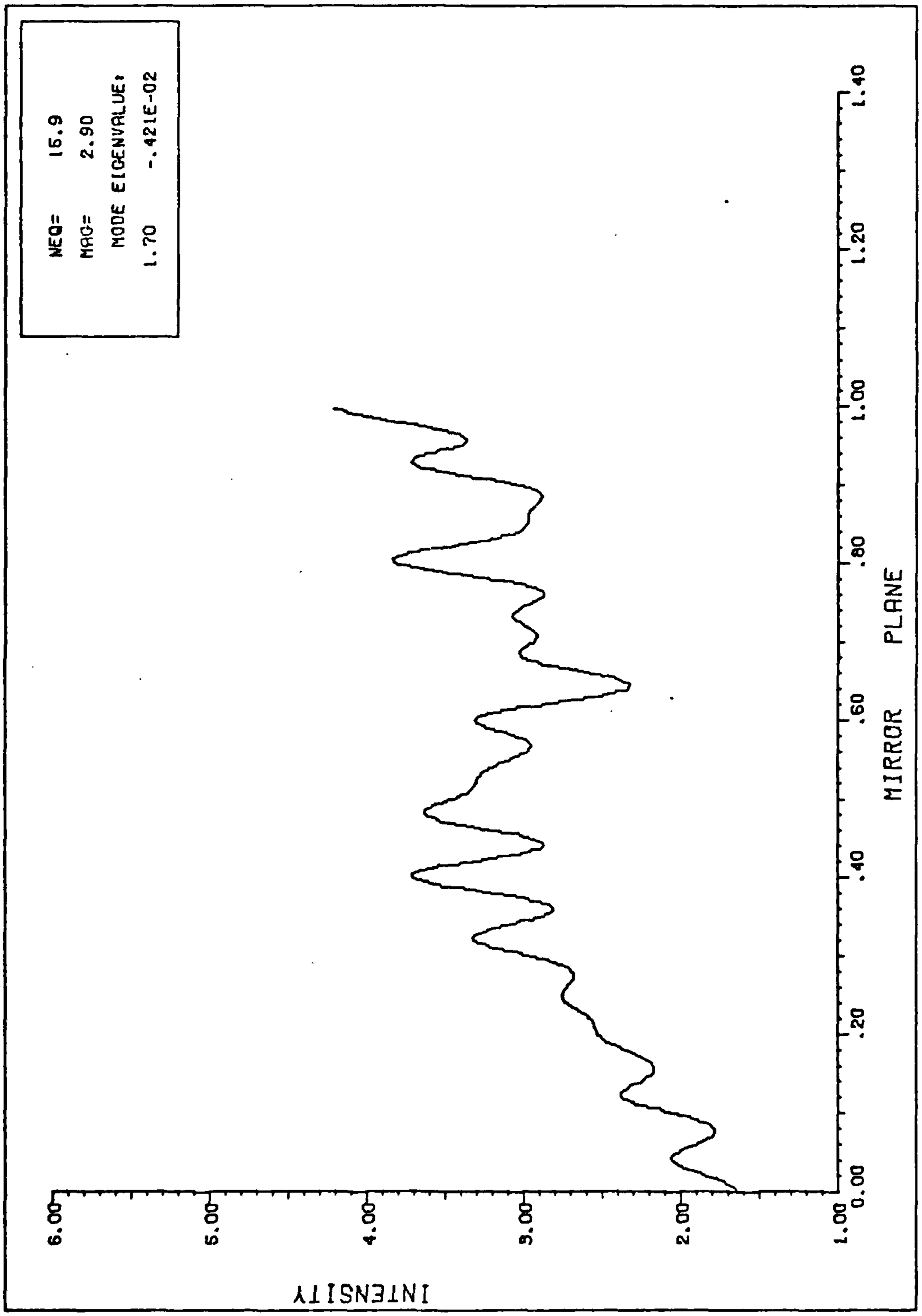
EVEN PARITY MODE # 1 APPROX # 1

Figure 21



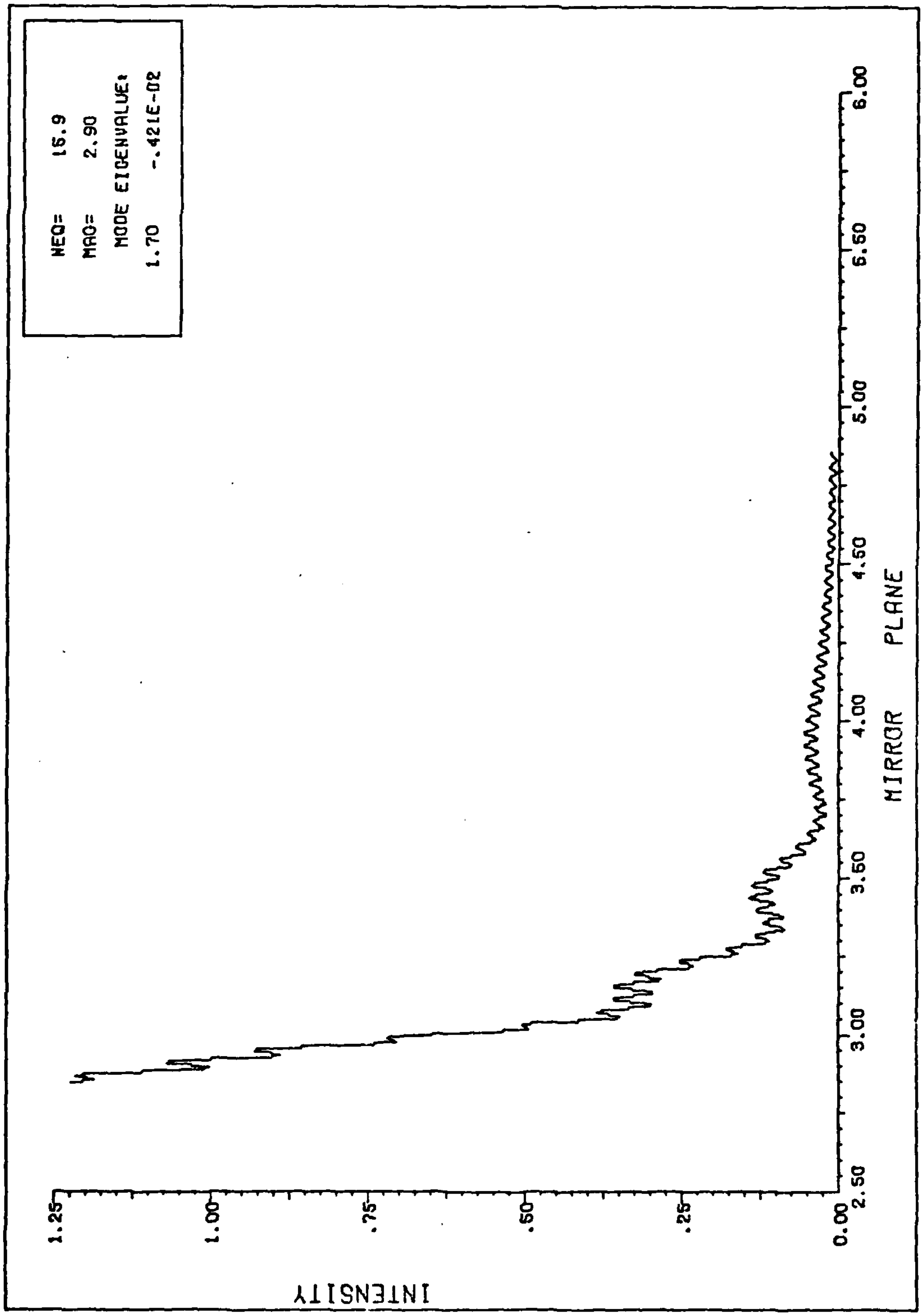
EVEN PARITY MODE # 1 APPROX # 2

Figure 22



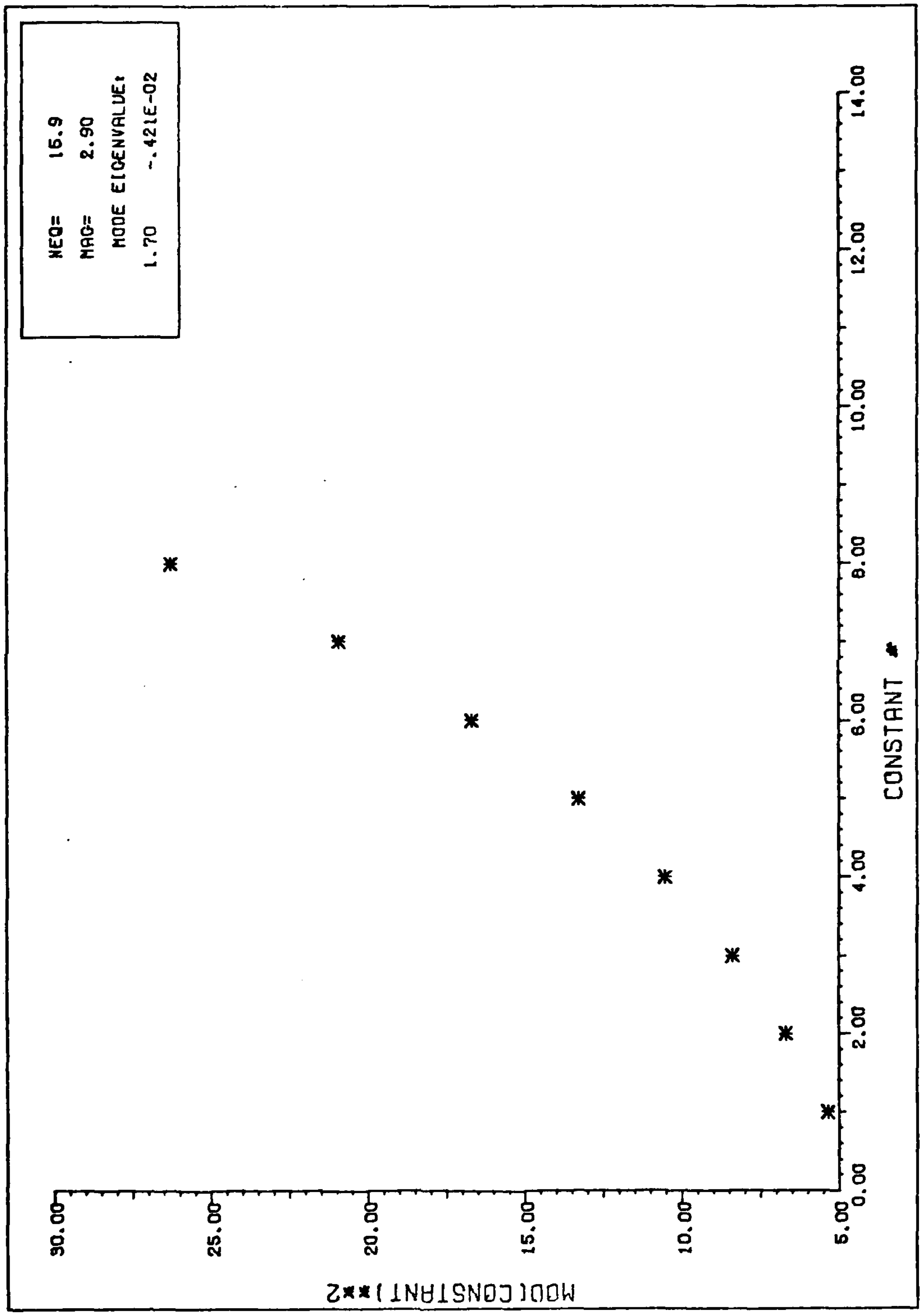
EVEN PARITY MODE # 1 APPROX # 1

Figure 23



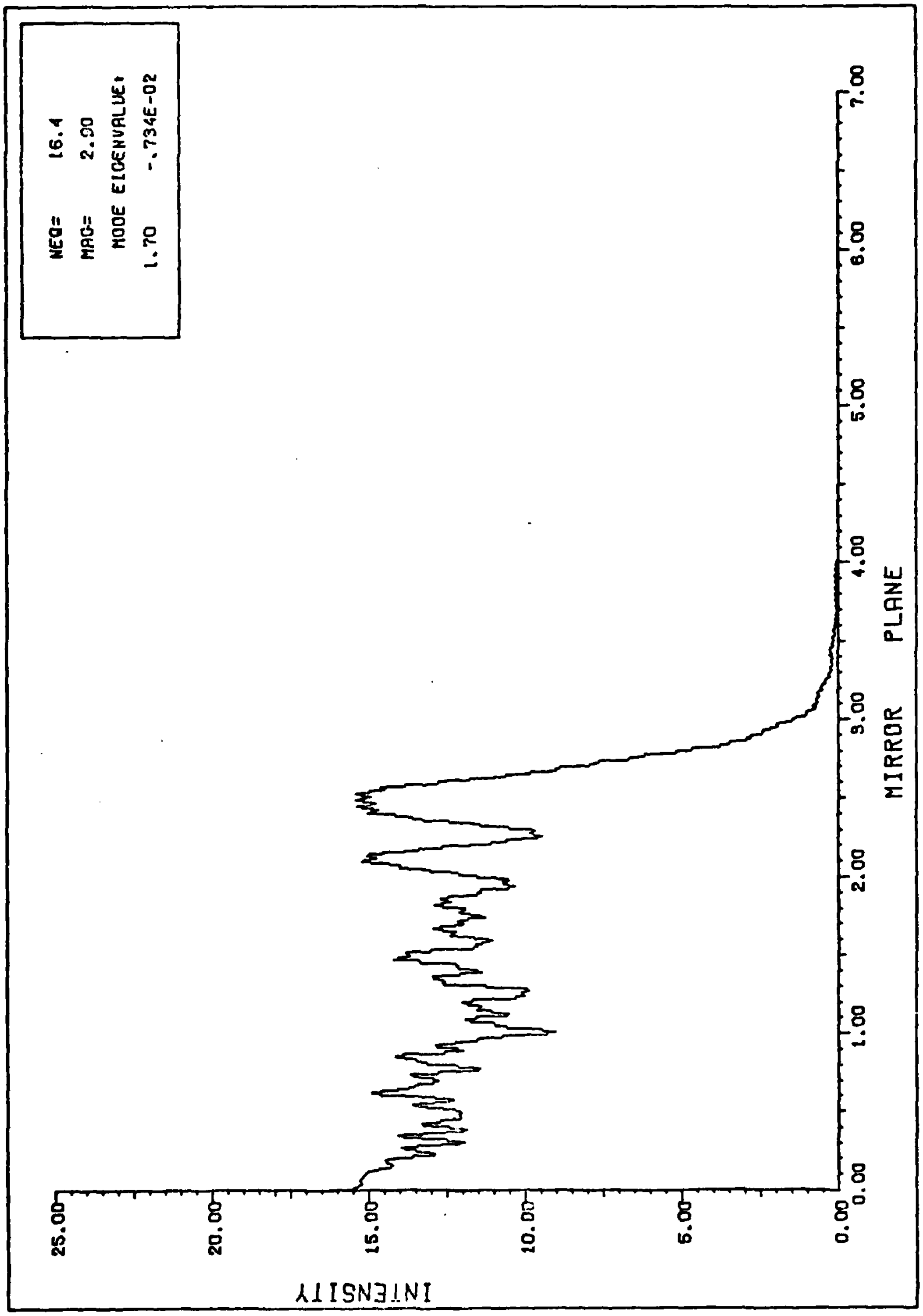
EVEN PARITY MODE # 1 APPROX # 2

Figure 24



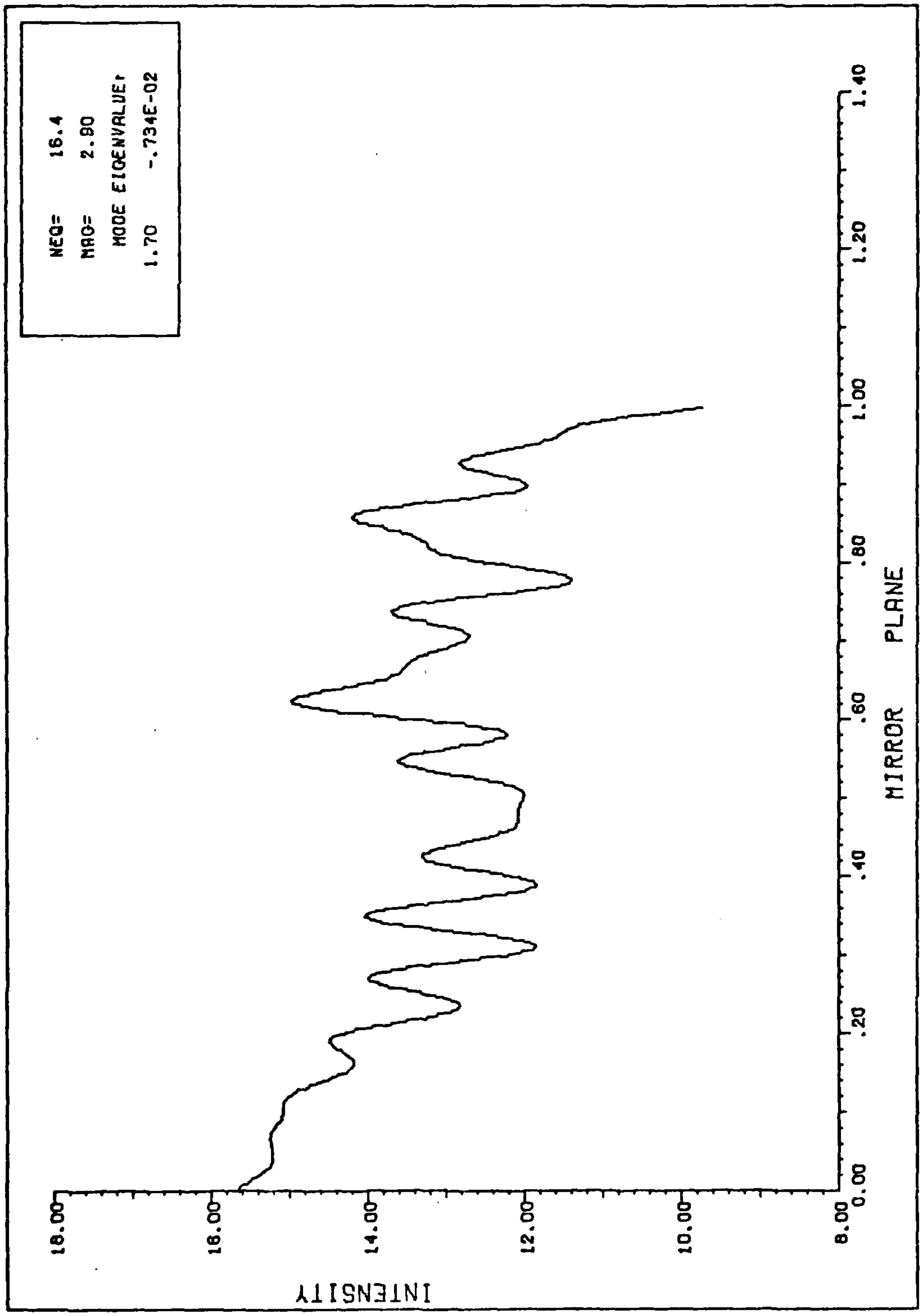
EVEN PARITY MODE # 1 APPROX # 1

Figure 25



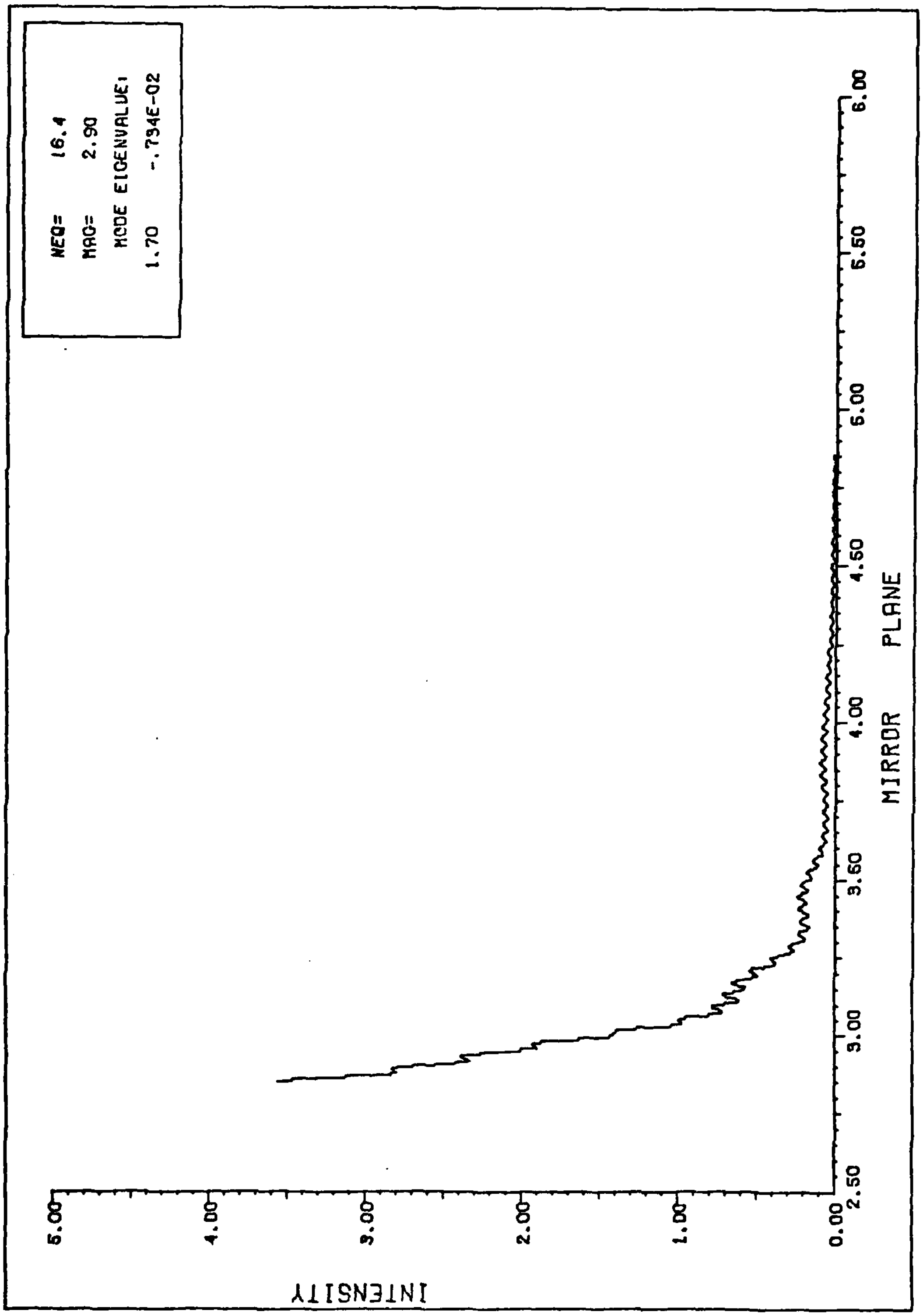
EVEN PARITY MODE # 1 APPROX # 2

Figure 26



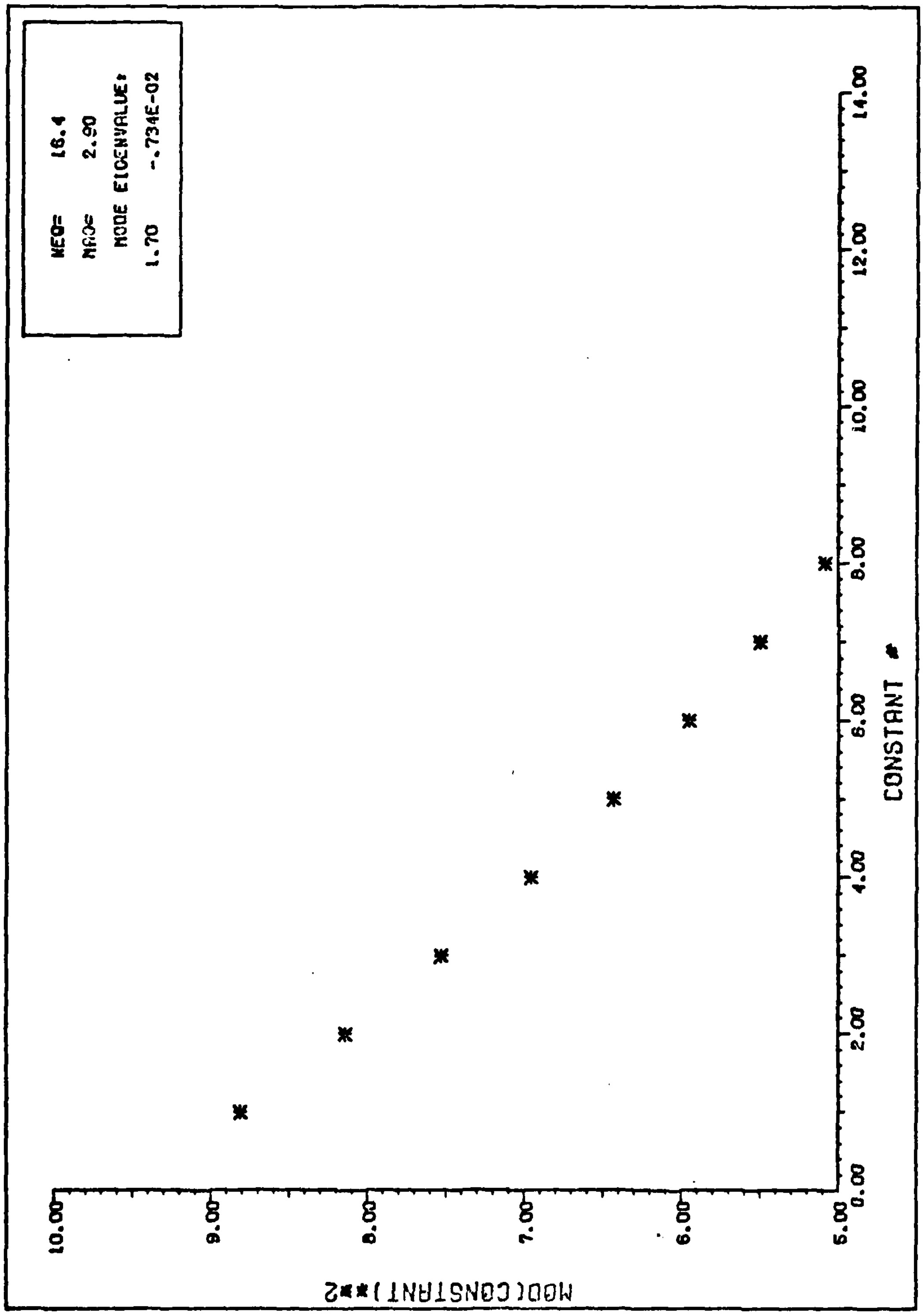
EVEN PARITY MODE # 1 APPROX # 1

Figure 27



EVEN PARITY MODE # 1 APPROX # 2

Figure 28



EVEN PARITY MODE # 1 APPROX # 1

Figure 29

In closer analysis, the mode for the bare resonator has eigenfunctions given by this expression

$$g^b(x) = 1+\sum_{n=1}^{N} c_n^b H_n(x) \qquad 5.2.1$$

while the loaded case has eigenfunctions given by

$$g^L(x) = h + \sum_{n=1}^{N} C_n{}^L H_n(x) \qquad 5.2.2$$

Similarly, the expressions for the weighting constants are

$$C_n{}^b = \frac{(\lambda-1)}{H_{N+1}} \lambda^{N-n} \qquad 5.2.3$$

$$C_n{}^L = \frac{(\lambda-\xi)}{H_{N+1}} h\left(\frac{\lambda}{\xi}\right)^{N-n} \frac{1}{\xi} \qquad 5.2.4$$

the eigenvalue polynomial in the loaded case is (4.1.32)

$$\lambda^N(\lambda-\xi) = \xi^{N+1}H_{N+1} + (\lambda-\xi)\sum_{n=1}^{N} \lambda^{N-n}\xi^n H_n(1) \qquad 5.2.5$$

Dividing through by ξ^{N+1} yields

$$\left(\frac{\lambda}{\xi}\right)^N\left(\frac{\lambda}{\xi}-1\right) = H_{N+1} + \left(\frac{\lambda}{\xi}-1\right)\sum_{n=1}^{N} \lambda^{N-n}H_n(1) \qquad 5.2.6$$

which becomes identical to the bare cavity polynomial (3.2.15)

as $\xi \to 1$. This condition will be fulfilled when h , the intensity ratio, becomes very large as seen from 4.1.19 and 4.1.4 . In turn H is then seen that, as $\xi \to 1$

$$C_n^{\ L} \to C_n^{\ b} \tag{5.2.7}$$

and

$$g^L(x) \to hg^b(x) \tag{5.2.8}$$

From this it is concluded that in the well saturated case, or when the ratio of the actual intensity to the saturation intensity is much more than one, the field distributions and hence the intensity profiles will equal those of the bare cavity case multiplied by h and h^2 respectively.

Tables 2 and 3 are presented to illustrate and compare mode separation properties of a loaded and a bare cavity for three different equivalent Fresnel numbers. The parameters chosen were a a magnification of 2.9 and N_f's of 16.874, 16.4 and 15.863. These were shown in reference 6 (Ref 6:1534) to be points of least, greatest and then least loss and next to lowest loss eigenvalue moduli. It was thought that since the lowest loss mode eigenvalue was forced to the same constant value at each Fresnel number, negating any quasiperiodicity, the higher loss modes might also lose quasi periodicity. The numbers presented show that the higher loss modes do maintain their quasi periodicity.

TABLE 2

BARE RESONATOR		Mod (λ)	
Mode	N_f=15.863	N_f=16.400	N_f=16.874
1	0.8543652	1.040102	0.8922496
2	0.8508141	0.6255715	0.7785354
3	0.5385818	0.6067205	0.5400256
4	0.5049350	0.4966156	0.5290538
5	0.4737932	0.4673182	0.4752758
6	0.1718837	0.1646309	0.1593562

TABLE 3

LOADED RESONATOR		Mod (λ)	
Mode	N_f=15.863	N_f=16.4	N_f=16.874
1.	1.702922	1.702965	1.70294
2	1.695844	1.024215	1.485906
3	1.073502	0.9933517	1.030688
4	1.006437	0.8130828	1.009747
5	0.8718129	0.7651157	0.9071073
6	0.3425989	0.2695415	0.3041459

The h's required to adjust λ to $\sim\sqrt{m}$ were 2.6899 3.1051, and 2.5979 for N_f=15.863, 16.4, and 16.874 respectively.

VI. Conclusion and Recommendations

Conclusion

The primary conclusion of this thesis is that program BARC, written according to expressions developed along Horwit'z analysis, produces valid results. The program allows analysis of even and odd parity mode solutions, more general than Moore and McCarthy's program, and also allows field calculation beyond the shadow boundary.

Incorporation of gain considerations into the program to allow analysis of a loaded strip resonator has been done. After modification the program produces results from which a second conclusion can be drawn, that being, for this particular model, mode intensity profiles in a loaded strip resonator are essentially the same as those predicted for a bare strip resonator. It is also concluded that mode losses as function of equivalent fresnel number continue to exhibit quasi periodicity in the loaded case.

Recommendations

The computer program, as it stands, predicts some very basic results about modes in an unstable resonator. There is no doubt that the scope of the program and the model upon which it is based can be broadened considerably. As it stands, it could be used to examine a full, or more complete

range of resonator parameters either loaded or bare.

The program should be used to explore mode separation in the loaded cavity case. Mode separation could be examined for a range of Fresnel numbers as has been done for bare cavities. (Ref 6)

The model given here could be modified to account for a non uniform gain function. To do this a new series of H_n's might be developed through asymptotic analysis of the gain-modified kernel. Another method might be the use of matrix methods to solve the eigenvalue equation.

This existing method could be applied to resonators with circular mirrors per Ref 2, and gain then included in that case.

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Appendix A

This appendix will employ the stationary phase approximation to simplify the resonator integral equation into a workable expression. The derivation starts with the functions.

$$F(x,t) = \frac{-1}{2\sqrt{i\pi t}} \frac{e^{-it(1-x)^2}}{1-x} \qquad \text{A1}$$

and

$$G(x,t) = \frac{-1}{2\sqrt{i\pi t}} \frac{e^{-it(1+x)^2}}{1+x} \qquad \text{A2}$$

These are modified by letting

$$F_n(x) = F\left(\frac{x}{m^n}, \frac{t}{m_{n-1}}\right) \qquad \text{A3}$$

and

$$G_n(x) = G\left(\frac{x}{m^n}, \frac{t}{m_{n-1}}\right) \qquad \text{A4}$$

where

$$m_n = \sum_{K=0}^{n} m^{-2K} \qquad \text{A5}$$

and

$$m = \text{magnification}$$

$$i = \sqrt{-1}$$

Thus it is seen that

$$F_1(x) = \frac{-1}{2\sqrt{i\pi t}} \frac{e^{-it(1-x/m)^2}}{1-x/m} \qquad \text{A6}$$

$$F_2(x) = \frac{-\sqrt{1+1/m^2}}{2\ i\ t} \frac{e^{-it(1-x/m^2)^2/1+1/m^2}}{1-x/m^2} \qquad \text{A7}$$

and in general that

$$F_n(x) = \frac{-\sqrt{m_{n-1}}}{2\sqrt{i\pi t}} \frac{e^{-it(1-x/m^n)^2/m_{n-1}}}{1-x/m^n} \qquad \text{A8}$$

and similarly

$$G_n(x) = \frac{-\sqrt{m_{n-1}}}{2\sqrt{i\pi t}} \frac{e^{-it(1+x/m^n)^2/m_{n-1}}}{1+x/m^n} \qquad \text{A9}$$

Now, it is recalled that the working form of the integral equation is

$$\lambda f(x) = \sqrt{\frac{it}{\pi}} \int_{-1}^{1} e^{-it(y-x/m)^2} f(y)\ dy \qquad \text{A10}$$

In the even part

$$f(x) = 1+ \sum_{n=1}^{N} (a_n F_n(x)+b_n G_n(x)) \qquad \text{A11}$$

A11 is substituted into the integral equation, which now becomes

$$\lambda(1+\Sigma\{a_n F_n(x)+b_n G_n(x)\}) =$$

$$\frac{it}{\pi} \int_{-1}^{1} e^{it(y-x/m)^2} (1+\Sigma\{a_n F_n(y)+b_n G_n(y)\})\ dy \qquad \text{A12}$$

When expanded once, the right side becomes

$$= \sqrt{\frac{it}{\pi}} \int_{-1}^{1} e^{-it(y-x/m)^2} dy + \sqrt{\frac{it}{\pi}} \sum_{n=1}^{N} \int_{-1}^{1} e^{-it(y-x/m)^2}$$

$$(a_n F_n(y)+b_n G_n(y)\ dy \qquad \text{A13}$$

The first term is called I_0 : Then

$$I_0 = \sqrt{\frac{it}{\pi}} \int_{-1}^{1} e^{-it(y-x/m)^2} dy \qquad \text{A14}$$

This is now considered in light of the first order approximation to the method of stationary phase which states that if

$$I = \int_a^b q(y)e^{-itp(y)}dy \qquad \text{A15}$$

then

$$I \simeq e^{-i\pi/4} q(y_0) e^{itp(y_0)} \sqrt{\frac{2\pi}{tp''(y_0)}}$$

$$+ \frac{i}{t}\left[\frac{q(b)}{p'(b)} e^{-itp(b)} - \frac{q(a)}{p'(a)} e^{-itp(a)}\right] \qquad \text{A16}$$

Where y_0 is such that

$$p'(y_0) = 0 \qquad \text{A17}$$

It is seen that

$$q(y) = 1 \qquad \text{A17.1}$$

$$p(y) = (y-x/m)^2 \qquad \text{A18}$$

$$p'(y) = 2(y-x/m) \qquad \text{A19}$$

$$p''(y) = 2 \qquad \text{A20}$$

And that

$$y_0 = x/m \qquad \text{A21}$$

Therefore, after substitution,

$$I_0 \simeq \sqrt{\frac{it}{\pi}}\left\{e^{-i\pi/4}\sqrt{\pi/t} + \frac{i}{t}\left[\frac{e^{-it(1-x/m)^2}}{2(1-x/m)} - \frac{e^{-it(-1-x/m)^2}}{2(-1-x/m)}\right]\right\} \quad A22$$

$$\simeq \sqrt{i}\, e^{-i\pi/4} + \frac{i}{2t}\sqrt{it/\pi}\left[\frac{e^{-it(1-x/m)^2}}{1-x/m} + \frac{e^{-it(1+x/m)^2}}{1+x/m}\right] \quad A23$$

However, since

$$e^{-i\pi/4}\sqrt{i} = \tfrac{1}{2}(1+i)(1-i) \quad A24$$

$$=1 \quad A25$$

Then

$$I_0 \simeq 1+\frac{i}{2t}\sqrt{it/\pi}\left[\frac{e^{-it(1-x/m)^2}}{1-x/m} + \frac{e^{-it(1+x/m)^2}}{1+x/m}\right] \quad A26$$

$$\simeq 1+\frac{i\sqrt{i}}{2\sqrt{\pi t}}\left[\frac{e^{-it(1-x/m)^2}}{1-x/m} + \frac{e^{-it(1+x/m)^2}}{1+x/m}\right] \quad A27$$

Since $i\sqrt{i} = (i-1)/\sqrt{2}$, and $-1/\sqrt{i} = (i-1)/\sqrt{2}$, it is seen that however, $i\sqrt{i} = \sqrt{i} = \frac{1}{q}$

$$I_0 \simeq 1-\frac{1}{2\sqrt{i\pi t}}\left[\frac{e^{-it(1-x/m)^2}}{1-x/m} + \frac{e^{-it(1+x/m)^2}}{1+x/m}\right] \quad A28$$

It is now easily seen that

$$I_0 \simeq 1+F_1(x)+G_1(x) \quad A29$$

The first term in the sum of integrals is now called I_1:

$$I_1 = \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-x/m)^2}(a_1F_1(y) + b_1G_1(y)) \; dy \qquad \text{A30}$$

Explicitly, this becomes

$$I_1 = \sqrt{it/\pi} \int_{-1}^{1} \left[\frac{-a_1}{2\sqrt{i\pi t}} \frac{e^{-it(1-y/m)^2}}{1-y/m} + \frac{-b_1}{2\sqrt{i\pi t}} \frac{e^{-it(1+y/m)^2}}{1+y/m} \right] \cdot e^{-it(y-x/m)^2} \qquad \text{A31}$$

Upon separation the result is

$$I = \sqrt{it/\pi} \int_{-1}^{1} \frac{-a_1}{2\sqrt{i\pi t}} \frac{e^{-it|(1-y/m)^2+(y-x/m)^2|}}{1-y/m} \; dy$$

$$+ \sqrt{it/\pi} \int_{-1}^{1} \frac{-b_1}{2\sqrt{i\pi t}} \frac{e^{-it|(1+y/m)^2+(y-x/m)^2|}}{1+y/m} \qquad \text{A32}$$

Upon consideration of the first part of this, it is seen that

$$q(y) = \frac{1}{1-y/m} \qquad \text{A33}$$

$$p(y) = (1-y/m)^2+(y-x/m)^2 \qquad \text{A34}$$

$$p'(y) = \frac{-2}{m}\left(1-\frac{y}{m}\right) + 2\left(y-\frac{x}{m}\right) \qquad \text{A35}$$

$$p''(y) = \frac{2}{m^2} + 2 \qquad \text{A36}$$

Solving for y_0 ,

$$0 = \frac{-2}{m} + \frac{2y_0}{m^2} + 2y_0 - \frac{2x}{m} \qquad \text{A37}$$

$$y_0\left(1+\frac{1}{m^2}\right) = \frac{1}{m} + \frac{x}{m} \qquad \text{A38}$$

$$y_0 = \frac{\left(\frac{1}{m} + \frac{x}{m}\right)}{1 + \frac{1}{m^2}} \qquad \text{A39}$$

These expressions are now substituted into the first part of A32 to get

$$\frac{-a_1}{2\sqrt{i\pi t}}\sqrt{it/\pi}\left\{\frac{e^{-it\left[\left(1-\frac{1}{m}\cdot\frac{1}{1+\frac{1}{m^2}}\cdot\left(\frac{1}{m}+\frac{x}{m}\right)\right)^2+\left(\left(\frac{1}{m}+\frac{x}{m}\right)\cdot\frac{1}{1+\frac{1}{m^2}}-\frac{x}{m}\right)^2\right]}}{1-\left(\frac{1}{m}\left(\frac{1}{1+\frac{1}{m^2}}\right)\left(\frac{1}{m}+\frac{x}{m}\right)\right)}\right.$$

$$\left.+\frac{i}{t}\left[\frac{e^{-it\left(\left(1-\frac{y}{m}\right)^2+\left(1-\frac{x}{m}\right)\right)^2}}{\left(1-\frac{1}{m}\right)\left(\frac{-2}{m}\left(1-\frac{1}{m}\right)+2\left(1-\frac{x}{m}\right)\right)} - \frac{e^{it\left(\left(1+\frac{1}{m}\right)^2+\left(-1-\frac{x}{m}\right)^2\right)}}{\left(1+\frac{1}{m}\right)\left(\frac{-2}{m}\left(1+\frac{1}{m}\right)+2\left(-1-\frac{x}{m}\right)\right)}\right]\right\} \qquad \text{A40}$$

Considering the second term of A32 shows that

$$q(y) = \frac{1}{1+y/m} \qquad \text{A41}$$

$$p(y) = (1+\frac{y}{m})^2+(y-\frac{x}{m})^2 \qquad \text{A42}$$

$$p'(y) = \frac{2}{m}(1+\frac{y}{m})+2(y-\frac{x}{m}) \qquad \text{A43}$$

$$p''(y) = \frac{2}{m^2} + 2 \qquad \text{A44}$$

and solving for y_0 , the result is

$$0 = \frac{2}{m} + \frac{2y_0}{m^2} + 2y_0 - \frac{2x}{m} \qquad \text{A45}$$

$$y_0(\frac{1}{m^2}+1) = \frac{x}{m} - \frac{1}{m} \qquad \text{A46}$$

$$y_0 = (\frac{x}{m} - \frac{1}{m})\cdot\frac{1}{1+\frac{1}{m^2}} \qquad \text{A47}$$

Substitution of these expressions into the second part of A32 results in

$$\frac{-b_1}{2\sqrt{i\pi t}}\sqrt{it/\pi}\left\{ e^{\frac{-i\pi}{4}}\sqrt{\frac{\pi}{t1+\frac{1}{m^2}}}\;\frac{e^{-it\left[1+\frac{1}{m}\left(\frac{1}{1+\frac{1}{m^2}}(\frac{x}{m}-\frac{1}{m})\right)^2+\left(\frac{1}{1+\frac{1}{m^2}}(\frac{x}{m}-\frac{1}{m})-\frac{x}{m}\right)^2\right]}}{1+\left(\frac{1}{m}\cdot\frac{1}{1+\frac{1}{m^2}}(\frac{x}{m}-\frac{1}{m})\right)} \right.$$

$$\left. +\ \frac{i}{t}\left[\frac{e^{-it((1+\frac{1}{m})^2+(1-\frac{x}{m})^2)}}{(1+\frac{1}{m})\cdot(\frac{2}{m}(1+\frac{1}{m})+2(1-\frac{x}{m}))} - \frac{e^{-it((1-\frac{1}{m})^2+(-1-\frac{x}{m})^2)}}{(1-\frac{1}{m})\cdot(\frac{2}{m}(1-\frac{1}{m})+2(-1-\frac{x}{m}))}\right]\right\} \qquad \text{A48}$$

Considering the denominator in (A-40)'s stationary phase point contribution, it is seen that

$$1-\frac{1}{m}\,\frac{1}{1+\frac{1}{m^2}}\left(\frac{1}{m}+\frac{x}{m}\right)=1-\frac{x+1}{(m+\frac{1}{m})\cdot m} \qquad \text{A49}$$

And similarly in (A-48)'s stat phase point cont, it is seen that

$$1+\frac{1}{m}\,\frac{1}{1+\frac{1}{m^2}}\left(\frac{x}{m}-\frac{1}{m}\right)=1-\frac{x-1}{(m+\frac{1}{m})\cdot m} \qquad \text{A50}$$

Also, it is seen that the argument of the exponent in A-40's stationary phase contribution can be simplified as follows

$$\left(1-\frac{1}{m}\,\frac{1}{1+\frac{1}{m^2}}\left(\frac{x}{m}+\frac{1}{m}\right)\right)^2+\left(\frac{1}{1+\frac{1}{m^2}}\left(\frac{1}{m}+\frac{x}{m}\right)-\frac{x}{m}\right)^2 = \qquad \text{A51}$$

$$\left(1-\frac{1}{m+\frac{1}{m}}\,\frac{x}{m}+\frac{1}{m}\right)^2+\left(\frac{x+1}{m+\frac{1}{m}}-\frac{x}{m}\right)^2 = \qquad \text{A52}$$

$$\left(1-\frac{x+1}{m^2+1}\right)^2+\left(\frac{x+1}{m+\frac{1}{m}}-\frac{x}{m}\right)^2 = \qquad \text{A53}$$

$$\left(\frac{m^2+1-x-1}{m^2+1}\right)^2+\left(\frac{m(x+1)-x(m+\frac{1}{m})}{m^2+1}\right)^2 = \qquad \text{A54}$$

$$\frac{(m^2-x)^2+(m(x+1)-x(m+\frac{1}{m}))^2}{(m^2+1)^2} = \qquad \text{A55}$$

$$= \frac{(1-\frac{x}{m^2})^2}{(1+\frac{1}{m^2})^2} + \frac{(1-\frac{x}{m^2})^2}{(m+\frac{1}{m})^2} \qquad \text{A56}$$

$$= \frac{(1-\frac{x}{m^2})^2}{(1+\frac{1}{m^2})^2} + \frac{\frac{1}{m^2}(1-\frac{x}{m^2})^2}{(1+\frac{1}{m^2})^2} \qquad \text{A57}$$

$$= \frac{(1-\frac{x}{m^2})(1+\frac{1}{m^2})}{(1+\frac{1}{m^2})} \qquad \text{A58}$$

$$= \frac{(1-\frac{x}{m^2})^2}{1+\frac{1}{m^2}} \qquad \text{A59}$$

In A48, the exp argument in the stationary phase point contribution can be simplified as follows

$$\left(1+\frac{1}{m}\,\frac{1}{1+\frac{1}{m^2}}\left(\frac{x}{m}-\frac{1}{m}\right)\right)^2+\left(\frac{1}{1+\frac{1}{m^2}}\left(\frac{x}{m}-\frac{1}{m}\right)-\frac{x}{m}\right)^2 \qquad \text{A60}$$

$$= \left(1+\frac{x-1}{m^2+1}\right)^2+\left(\frac{x-1}{m+\frac{1}{m}}-\frac{x}{m}\right)^2 \qquad \text{A61}$$

$$= \left(\frac{m^2+1+x-1}{m^2+1}\right)^2+\left(\frac{mx-m-mx-\frac{x}{m}}{m^2+1}\right)^2 \qquad \text{A62}$$

$$= \left(\frac{m^2+x}{m^2+1}\right)^2+\left(\frac{-\frac{x}{m}-m}{m^2+1}\right)^2 \qquad \text{A63}$$

$$= \frac{(1+\frac{x}{m^2})^2}{(1+\frac{1}{m^2})} + \frac{(\frac{x}{m^2}+1)^2}{(\frac{m^2+1}{m})^2} \qquad \text{A64}$$

$$= \frac{(1+\frac{x}{m^2})^2}{(1+\frac{1}{m^2})^2} + \frac{\frac{1}{m^2}(\frac{x}{m^2}+1)^2}{(1+\frac{1}{m^2})^2} \qquad \text{A65}$$

$$= \frac{(\frac{1}{m^2}+1)(1+\frac{x}{m^2})^2}{(1+\frac{1}{m^2})^2} = \frac{(1+\frac{x}{m^2})^2}{1+\frac{1}{m^2}} \qquad \text{A66}$$

Substituting these simplified expressions into A-32, we find that

$$I_1 \simeq -\sqrt{it/\pi}\,\frac{1}{2\sqrt{i\pi t}}\left\{e^{-i\pi/4}\sqrt{\frac{\pi}{t(1+\frac{1}{m^2})}}\left(\frac{a_1 e^{-it(1-\frac{x}{m^2})^2/1+\frac{1}{m^2}}}{1-\frac{x+1}{m(m+\frac{1}{m})}}\right.\right.$$

$$\left.\left.+\frac{b_1 e^{-it(1+\frac{x}{m})^2/1+\frac{1}{m^2}}}{1+\frac{x-1}{m(m+\frac{1}{m})}}\right)\right\} - \sqrt{it/\pi}\,\frac{1}{2\sqrt{i\pi t}}\,\frac{i}{t}$$

$$\left\{\frac{a_1 e^{-it((1-\frac{1}{m})^2+(1-\frac{x}{m})^2)}}{(1-\frac{1}{m})(\frac{-2}{m}(1-\frac{1}{m})+2(1-\frac{x}{m}))} - \frac{a_1 e^{-it((1+\frac{1}{m})^2+(-1-\frac{x}{m})^2)}}{(1+\frac{1}{m})(\frac{-2}{m}(1+\frac{1}{m})+2(-1-\frac{x}{m}))}\right.$$

$$\left.+\frac{b_1 e^{-it(1+\frac{1}{m})^2+(1-\frac{x}{m})^2}}{(1+\frac{1}{m})(\frac{2}{m}(1+\frac{1}{m})+2(1-\frac{x}{m}))} - \frac{b_1 e^{-it(1-\frac{1}{m})^2+(-1-\frac{x}{m})^2}}{(1-\frac{1}{m})(\frac{2}{m}(1-\frac{1}{m})+2(-1-\frac{x}{m}))}\right\} \qquad \text{A67}$$

Now the first part of this expression can be simplified as follows:

$$\frac{1}{1-\frac{x+1}{m(m+\frac{1}{m})}} = \frac{1}{1-\frac{x+1}{m^2+1}} \qquad \text{A68}$$

$$= \frac{1}{\frac{m^2+1-x-1}{m^2+1}} \tag{A69}$$

$$= \frac{m^2+1}{m^2-x} \tag{A70}$$

$$= \frac{1+\frac{1}{m^2}}{1-\frac{x}{m^2}} \tag{A71}$$

And the second part can also be simplified

$$\frac{1}{1+\frac{x-1}{m^2+1}} = \frac{1}{\frac{m^2+1+x-1}{m^2+1}} \tag{A72}$$

$$= \frac{m^2+1}{m^2+x} \tag{A73}$$

$$= \frac{1+\frac{1}{m^2}}{1+\frac{x}{m^2}} \tag{A74}$$

If these simplified expressions are substituted into A67, the result is

$$I_1 \simeq -\sqrt{it/\pi}\ \frac{1}{2\sqrt{i\pi t}}\ e^{-i\pi/4} \sqrt{\frac{\pi}{t(1+\frac{1}{m^2})}} \left(\frac{a_1 e^{-it(1-\frac{x}{m^2})^2/1+\frac{1}{m^2}}}{1-\frac{x}{m^2}} + \frac{b_1 e^{-it(1+\frac{x}{m^2})^2/1+\frac{1}{m^2})}}{1+\frac{x}{m^2}} \right) - \sqrt{it/\pi}\ \frac{1}{2\sqrt{i\pi t}}\ \frac{1}{2t}$$

$$\left\{ \frac{a_1 e^{-it((1-\frac{1}{m})^2+(1-\frac{x}{m})^2)}}{(1-\frac{1}{m})(-\frac{1}{m}+\frac{1}{m^2}+1-\frac{x}{m})} - \frac{a_1 e^{-it((1+\frac{1}{m})^2+(-1-\frac{x}{m})^2)}}{(1+\frac{1}{m})(-\frac{1}{m}-\frac{1}{m^2}-1-\frac{x}{m})} \right.$$

$$\left. + \frac{b_1 e^{-it((1+\frac{1}{m})^2+(1-\frac{x}{m})^2)}}{(1+\frac{1}{m})(\frac{1}{m}+\frac{1}{m^2}+1-\frac{x}{m})} - \frac{b_1 e^{-it((1-\frac{1}{m})^2+(-1-\frac{x}{m})^2)}}{(1-\frac{1}{m})(\frac{1}{m}-\frac{1}{m^2}-1-\frac{x}{m})} \right\} \qquad \text{A75}$$

After a few sign manipulations and cancellations, it is seen that

$$I_1 \simeq \frac{-1}{2\sqrt{i\pi t}} \sqrt{1+1/m^2} \left(\frac{a_1 e^{-it(1-\frac{x}{m^2})^2/1+\frac{1}{m^2}}}{1-\frac{x}{m^2}} + \frac{b_1 e^{-it(1+\frac{x}{m^2})^2/1+\frac{1}{m_2}}}{1+\frac{x}{m^2}} \right)$$

$$+ \frac{1}{4i\pi t} \left\{ \frac{a_1 e^{-it(1-\frac{1}{m})^2} e^{-it(1-\frac{x}{m})^2}}{(1-\frac{1}{m})(1-\frac{1}{m}+\frac{1}{m^2}-\frac{x}{m})} + \frac{a_1 e^{-it(1+\frac{1}{m})^2} e^{-it(-1-\frac{x}{m})^2}}{(1+\frac{1}{m})(1+\frac{1}{m}+\frac{1}{m^2}+\frac{x}{m})} \right.$$

$$\left. + \frac{b_1 e^{-it(1+\frac{1}{m})^2} e^{-it(1-\frac{x}{m})^2}}{(1+\frac{1}{m})(1+\frac{1}{m}+\frac{1}{m^2}-\frac{x}{m})} + \frac{b_1 e^{-it(1-\frac{1}{m})^2} e^{-it(-1-\frac{x}{m})^2}}{(1-\frac{1}{m})(1-\frac{1}{m}+\frac{1}{m^2}+\frac{x}{m})} \right\} \qquad \text{A76}$$

The stationary phase point contributions are seen immediately to be equal to

$$a_1F_2(x)+b_1G_2(x) \qquad \text{A77}$$

In the end point contributions, the denominators must be approximated and terms of $\frac{1}{m}$ or higher order be neglected. When done, the end point contributions appear as

$$+\frac{1}{4i\pi t}\left\{\frac{a_1 e^{-it(1-\frac{1}{m})^2} e^{-it(1-\frac{x}{m})^2}}{(1-\frac{1}{m})(1-\frac{x}{m})}+\frac{a_1 e^{-it(1+\frac{1}{m})^2} e^{-it(1+\frac{x}{m})^2}}{(1+\frac{1}{m})(1+\frac{x}{m})}\right.$$

$$\left.+\frac{b_1 e^{-it(1+\frac{1}{m})^2} e^{-it(1-\frac{x}{m})^2}}{(1+\frac{1}{m})(1-\frac{x}{m})}+\frac{b_1 e^{-it(1-\frac{1}{m})^2} e^{-it(1+\frac{x}{m})^2}}{(1+\frac{x}{m})(1-\frac{1}{m})}\right\} \qquad \text{A78}$$

If

$$\frac{1}{4i\pi t}=\left(\frac{-1}{2\sqrt{i\pi t}}\right)^2 \qquad \text{A79}$$

then these contributions are approximately equal to

$$a_1F_1(x)F_1(1)+a_1G_1(x)F_1(-1)+b_1F_1(x)G_1(1)+b_1G_1(x)G_1(-1) \qquad \text{A80}$$

And finally,

$$\begin{aligned}I_1 \simeq\ & a_1F_2(x)+b_1G_2(x)\\ &+a_1(F_1(x)F_1(1)+G_1(x)F_1(-1))\\ &+b_1(F_1(x)G_1(1)+G_1(x)G_1(-1))\end{aligned} \qquad \text{A81}$$

The second term in the sum of integrals is now considered as I_2 :

$$I_2 = \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} (a_2F_2(y)+b_2G_2(y))\ dy$$

$$= \sqrt{it/\pi} \int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} \left\{ \frac{-a_2\ 1+\frac{1}{m^2}}{2\sqrt{i\pi t}}\ \frac{e^{-it(1-\frac{y}{m^2})^2/1+\frac{1}{m^2}}}{1-\frac{y}{m^2}} \right.$$

$$\left. - \frac{b_2\ 1+\frac{1}{m^2}}{2\sqrt{i\pi t}}\ \frac{e^{-it(1+\frac{y}{m^2})^2/1+\frac{1}{m^2}}}{1+\frac{y}{m^2}} \right\} dy \qquad \text{A82}$$

Considering the first term of this expression, it is seen that

$$q(y) = \frac{1}{1-\frac{y}{m^2}} \qquad \text{A83}$$

$$p(y) = (y-\frac{x}{m})^2 + \frac{1}{1+\frac{1}{m^2}}(1-\frac{y}{m^2})^2 \qquad \text{A84}$$

$$p'(y) = 2(y-\frac{x}{m}) - \frac{2}{m^2(1+\frac{1}{m^2})}(1-\frac{y}{m^2}) \qquad \text{A85}$$

$$p''(y) = 2+\frac{2}{m^4(1+\frac{1}{m^2})} \qquad \text{A86}$$

And solving for y_0 it is seen that

$$0 = y_0-\frac{x}{m}-\left(\frac{1}{m^2(1+\frac{1}{m^2})}\right)(1-\frac{y_0}{m^2}) \qquad \text{A87}$$

$$0 = y_0 - \frac{x}{m} - \frac{1}{m^2(1+\frac{1}{m^2})} + \frac{y_0}{m^4(1+\frac{1}{m^2})} \qquad \text{A88}$$

$$y_0\left(1+\frac{1}{m^4(1+\frac{1}{m^2})}\right) = \frac{x}{m} + \frac{1}{m^2(1+\frac{1}{m^2})} \qquad \text{A89}$$

and

$$y_0 = \left(\frac{x}{m} + \frac{1}{m^2(1+\frac{1}{m^2})}\right)\left(\frac{1}{1+\frac{1}{m^4(1+\frac{1}{m^2})}}\right) \qquad \text{A90}$$

Substituting these expressions into A82, the result is that the first part of I_2 is

$$\frac{a_2}{2\sqrt{i\pi t}} \sqrt{it/\pi}\sqrt{1+1/m^2} \left[\frac{e^{\frac{-i\pi}{4}}\, e^{-it\left((y-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1-y_0)^2\right)}}{1-\frac{y_0}{m^2}}\right.$$

$$\left.\cdot\sqrt{\frac{\pi}{t\cdot 1+\left(\frac{1}{m^4(1+\frac{1}{m^2})}\right)}}\right] - \frac{a_2}{2\sqrt{i\pi t}} \sqrt{it/\pi}\ \sqrt{1+1/m^2}\ \frac{i}{t}$$

$$\cdot\left[\frac{e^{-it(1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2}}{(1-\frac{1}{m^2})\left(2(1-\frac{x}{m})-\frac{2}{m^2(1+\frac{1}{m^2})}(1-\frac{1}{m^2})\right)}\right.$$

$$\left.-\frac{e^{-it(-1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2}}{(1+\frac{1}{m^2})\left(2(-1-\frac{x}{m})-\frac{2}{m^2(1+\frac{1}{m^2})}(+\frac{1}{m^2})\right)}\right] \qquad \text{A91}$$

Considering the second part, it is seen that

$$q(y) = \frac{1}{1+\frac{y}{m^2}} \tag{A92}$$

$$p(y) = (y-\frac{x}{m^2})^2 + \frac{1}{1+\frac{1}{m^2}} (1+\frac{y}{m^2})^2 \tag{A93}$$

$$p'(y) = 2(y-\frac{x}{m}) + \frac{2}{1+\frac{1}{m^2}} \frac{1}{m^2}(1+\frac{y}{m^2}) \tag{A94}$$

$$p''(y) = 2+ \frac{2}{m^4(1+\frac{1}{m^2})} \tag{A95}$$

Solving for y_0 ,

$$0 = y_0-\frac{x}{m}+\frac{1}{m^2} \frac{1}{1+\frac{1}{m^2}}(1+\frac{y_0}{m^2}) \tag{A96}$$

$$0 = y_0-\frac{x}{m}+\frac{1}{m^2} \frac{1}{1+\frac{1}{m^2}} + \frac{y_0}{m^4(1+\frac{1}{m^2})} \tag{A97}$$

$$y_0\left(1+\frac{1}{m^4(1+\frac{1}{m^2})}\right) = \frac{x}{m} - \frac{1}{m^2(1+\frac{1}{m^2})} \tag{A98}$$

and finally

$$y_0 = \left(\frac{x}{m} - \frac{1}{m^2(1+\frac{1}{m^2})}\right)\left(\frac{1}{1 + \frac{1}{m^4(1+\frac{1}{m^2})}}\right) \tag{A99}$$

Substituting these expressions into A82 it is seen that the second part of I_2 is approximated by

$$\frac{-b_2}{2\sqrt{i\pi t}}\sqrt{it/\pi}\ \sqrt{1+1/m^2}\left[e^{-i\pi/4}\sqrt{\frac{\pi}{t(1+\frac{1}{m^4(1+\frac{1}{m^2})})}}\right.$$

$$\left.\frac{e^{-it(y_0\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1+\frac{y_0}{m^2})^2}}{1+\frac{y_0}{m^2}}\right]\frac{-b_2}{2\sqrt{i\pi t}}\sqrt{it/\pi}\ \sqrt{1+\frac{1}{m^2}}\ \frac{i}{t}$$

$$\left\{\frac{e^{-it((1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2}))^2}}{(1+\frac{1}{m^2})(2(1-\frac{x}{m})+\frac{2}{m^2}\ \frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2}))}-\frac{e^{-it((-1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2)}}{(1-\frac{1}{m^2})(2(-1-\frac{x}{m})+\frac{2}{m^2}\ \frac{1}{1+\frac{1}{m^2}}(1-\frac{1}{m^2}))}\right\}$$

A100

Combining the first and second terms the result is the complete approximation to I_2 , or

$$I_2 \simeq -\sqrt{1+1/m^2}\ \sqrt{it/\pi}\ \frac{1}{2\sqrt{i\pi t}}\sqrt{\frac{\pi}{t(1+\frac{1}{m^4(1+\frac{1}{m^2})})}}$$

$$\left\{\frac{a_2 e^{\frac{-i\pi}{4}}\ e^{-it((y\ -\frac{x}{m})^2+(1-\frac{y_0}{m^2})^2/1+\frac{1}{m^2})}}{1-\frac{y_0}{m^2}}\right.$$

$$+\left.\frac{b_2 e^{\frac{-i\pi}{4}} e^{-it((y_0-\frac{x}{m})^2+(1+\frac{y_0}{m^2})^2/1+\frac{1}{m^2})}}{1+\frac{y_0}{m^2}}\right\} \quad -\sqrt{1+1/m^2}\ \sqrt{it/\pi}$$

$$\frac{1}{2\sqrt{i\pi t}}\ \frac{i}{t}\left\{\frac{a_2 e^{-it((1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1-\frac{1}{m^2}))^2}}{(1-\frac{1}{m^2})(2(1-\frac{x}{m})-\frac{2}{m^2(1+\frac{1}{m^2})}(1-\frac{1}{m^2})}\right.$$

$$-\frac{a_2 e^{-it((-1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2)}}{(1+\frac{1}{m^2})(2(-1-\frac{x}{m})-\frac{2}{m^2(1+\frac{1}{m^2})}(1+\frac{1}{m^2}))}$$

$$+\frac{b_2 e^{-it((1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2}))}}{(1+\frac{1}{m^2}(2(1-\frac{x}{m})+\frac{2}{m^2}\ \frac{1}{1+\frac{1}{m^2}}(1+\frac{1}{m^2}))}$$

$$\left.-\frac{b_2 e^{-it((-1-\frac{x}{m})^2+\frac{1}{1+\frac{1}{m^2}}(1-\frac{1}{m^2}))}}{(1-\frac{1}{m^2})(2(-1-\frac{x}{m})+\frac{2}{m^2(1+\frac{1}{m^2})}(1-\frac{1}{m^2}))}\right\} \qquad \text{A101}$$

The denominators of the stationary phase point contribution terms must be simplified:

In the first term,

$$1-\frac{1}{m^2}\left(\frac{x}{m}+\frac{1}{m^2(1+\frac{1}{m^2})}\right)\left(\frac{1}{1+\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}}\right) \qquad \text{A102}$$

$$= 1 - \frac{1}{m^2}\left(\frac{x}{m} + \frac{1}{m^2+1}\right)\left(\frac{1}{1+\left(\frac{1}{m^4}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right)}\right) \qquad \text{A103}$$

$$= 1 - \left(\frac{x}{m^3} + \frac{1}{m^4+m^2}\right)\frac{1}{1+\frac{1}{m^4+m^2}} \qquad \text{A104}$$

$$= \frac{1 + \frac{1}{m^4+m^2} - \frac{x}{m^3} - \frac{1}{m^4+m^2}}{1 + \frac{1}{m^4+m^2}} \qquad \text{A105}$$

$$= \left(1 - \frac{x}{m^3}\right)\frac{1}{1 + \left(\frac{1}{m^4}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right)} \qquad \text{A106}$$

And in the second term,

$$1 + \frac{1}{m^2}\left(\frac{x}{m} - \left(\frac{1}{m^2}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right)\right)\left(\frac{1}{1+\frac{1}{m^4}\left(\frac{1}{1+\frac{1}{m^2}}\right)}\right) \qquad \text{A107}$$

$$= \left(1 + \frac{x}{m^3} - \frac{1}{m^4+m^2}\right)\frac{1}{1+\frac{1}{m^4+m^2}} \qquad \text{A108}$$

$$= \left(1 + \frac{1}{m^4+m^2} + \frac{x}{m^3} - \frac{1}{m^4+m^2}\right)\frac{1}{1+\frac{1}{m^4+m^2}} \qquad \text{A109}$$

$$= \left(1 + \frac{x}{m^3}\right)\frac{1}{1+\frac{1}{m^4}\left(\frac{1}{1+\frac{1}{m^2}}\right)} \qquad \text{A110}$$

Now, considering the argument for the exponent in the first part, it is seen that

$$\left(y_0-\frac{x}{m}\right)^2+\frac{\left(1-\frac{y_0}{m}\right)^2}{1+\frac{1}{m^2}} = \left(\left(\frac{x}{m}+\left(\frac{1}{m^2}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right)\right)\left(\frac{1}{1+\frac{1}{m^4}\frac{1}{1+\frac{1}{m^2}}}\right)-\frac{x}{m}\right)^2$$

$$+\frac{\left(1-\frac{1}{m^2}\left(\frac{x}{m}+\frac{1}{m^2}\,\frac{1}{1+\frac{1}{m^2}}\right)\frac{1}{1+\frac{1}{m^4}\,\frac{1}{1+\frac{1}{m^2}}}\right)^2}{1+\frac{1}{m^2}} \qquad \text{A111}$$

$$= \left(\left(\left(\frac{x}{m}+\frac{1}{m^2+1}\right)\frac{1}{1+\frac{1}{m^4+m^2}}\right)-\frac{x}{m}\right)^2+\frac{1}{1+\frac{1}{m^2}}\left(1-\left(\frac{x}{m^3}+\frac{1}{m^4+m^2}\right)\frac{1}{1+\frac{1}{m^4+m^2}}\right)^2 \qquad \text{A112}$$

$$= \left(\left(\frac{x}{m}+\frac{1}{m^2+1}\right)\frac{1}{1+\frac{1}{m^4+m^2}}-\frac{x}{m}\right)^2+\frac{1}{1+\frac{1}{m^2}}\left(\frac{1+\frac{1}{m^4+m^2}-\frac{x}{m^3}-\frac{1}{m^4+m^2}}{1+\frac{1}{m^4+m^2}}\right)^2 \qquad \text{A113}$$

$$= \left(\left(\frac{x}{m}+\frac{1}{m^2}\,\frac{1}{1+\frac{1}{m^2}}\right)\frac{1}{1+\frac{1}{m^4}\,\frac{1}{1+\frac{1}{m^2}}}-\frac{x}{m}\right)^2+\frac{1}{1+\frac{1}{m^2}}\left(1-\frac{1}{m^2}\left(\frac{x}{m}+\frac{1}{m^2}\,\frac{1}{1+\frac{1}{m^2}}\right)\frac{1}{1+\frac{1}{m^4}\,\frac{1}{1+\frac{1}{m^2}}}\right)^2$$

A113.1

The first half of A113 is

$$\left(\frac{\frac{x}{m}\left(1+\frac{1}{m^2}\right)+\frac{1}{m^2}}{1+\frac{1}{m^2}}\;\frac{1}{1+\frac{1}{m^4}\,\frac{1}{1+\frac{1}{m^2}}}-\frac{x}{m}\right)^2 \qquad \text{A114}$$

$$= \left(\left(\frac{x}{m} + \frac{x}{m^3} + \frac{1}{m^2}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right) \frac{1+\frac{1}{m^2}}{1+\frac{1}{m^2}+\frac{1}{m^4}} - \frac{x}{m} \right)^2 \qquad \text{A115}$$

$$= \left(\left(\frac{x}{m} + \frac{x}{m^3} + \frac{1}{m^2}\right)\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right) - \frac{x}{m} \right)^2 \qquad \text{A116}$$

$$= \left(\frac{x}{m} + \frac{x}{m^3} + \frac{1}{m^2} - \frac{x}{m}\left(1+\frac{1}{m^2}+\frac{1}{m^4}\right)\right)^2 \left(\frac{1}{1+1/m^2+1/m^4}\right)^2 \qquad \text{A117}$$

$$= \left(\frac{1}{m^2} - \frac{x}{m^5}\right)^2 \left(\frac{1}{1+1/m^2+1/m^4}\right)^2 \qquad \text{A118}$$

$$= \frac{1}{m^4} \left(1-\frac{x}{m^3}\right) \left(\frac{1}{1+1/m^2+1/m^4}\right)^2 = \text{first half} \qquad \text{A119}$$

And the second half is

$$\left(1 - \frac{1}{m^2} \left(\frac{x}{m} + \frac{1}{m^2} \frac{1}{1+1/m^2}\right) \frac{1}{1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}}\right)^2 \frac{1}{1+\frac{1}{m^2}} \qquad \text{A120}$$

$$= \frac{1}{1+\frac{1}{m^2}} \left(1+\frac{1}{m^4 \, 1+\frac{1}{m^2}} - \frac{1}{m^2}\left(\frac{x}{m} + \frac{1}{m^2} \frac{1}{1+\frac{1}{m^2}}\right)\right)^2 \left(\frac{1}{1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}}\right)^2 \qquad \text{A121}$$

$$= \frac{1}{1+\frac{1}{m^2}} \left(1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}} - \frac{x}{m^3} - \frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}\right)^2 \left(\frac{1}{1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}}\right)^2 \qquad \text{A122}$$

$$= \frac{1}{1+\frac{1}{m^2}} \left(1 - \frac{x}{m^3}\right)\left(\frac{1+1/m^2}{1+1/m^2+1/m^4}\right)^2 \qquad \text{A123}$$

$$= 1+\frac{1}{m^2} \left(\frac{1 - \frac{x}{m^3}}{1 + \frac{1}{m^2} + \frac{1}{m^4}}\right)^2 \qquad \text{A124}$$

the total argument is

$$\frac{1}{m^4} \left(1-\frac{x}{m^3}\right)\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 + \left(1+\frac{1}{m^2}\right)\left(1-\frac{x}{m^3}\right)^2\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A125}$$

$$= \left(1+\frac{1}{m^2} + \frac{1}{m^4}\right)\left(\frac{1-\frac{x}{m^3}}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A126}$$

$$= \frac{\left(1- \frac{x}{m^3}\right)^2}{1+\frac{1}{m^2}+\frac{1}{m^4}} \qquad \text{A127}$$

Considering the argument for the exponent in the second half,

$$\left(\left(\frac{x}{m} - \frac{1}{m^2} \cdot \frac{1}{1+\frac{1}{m^2}}\right)\left(\frac{1}{1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}}\right)\right)^2$$

$$+ \frac{1}{1+\frac{1}{m^2}}\left(1 + \frac{1}{m^2}\left(\frac{x}{m} - \frac{1}{m^2} \frac{1}{1+\frac{1}{m^2}}\right)\left(\frac{1}{1+\frac{1}{m^4} \frac{1}{1+\frac{1}{m^2}}}\right)\right)^2 \qquad \text{A128}$$

Considering the first part of this,

$$\frac{\frac{x}{m}\ 1+\frac{1}{m^2} - \frac{1}{m^2}}{1+\frac{1}{m^2}} \quad \frac{1}{1+\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}} - \frac{x}{m} \qquad \text{A129}$$

$$= \frac{\frac{x}{m} + \frac{x}{m^3} - \frac{1}{m^2}}{1+\frac{1}{m^2}} \quad \frac{1+\frac{1}{m^2}}{1+\frac{1}{m^2}+\frac{1}{m^4}} - \frac{x}{m} \qquad \text{A130}$$

$$= \left(\frac{x}{m} + \frac{x}{m^3} - \frac{1}{m^2} - \frac{x}{m}\left(1+\frac{1}{m^2} + \frac{1}{m^4}\right)\right)^2 \left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A131}$$

$$= \left(\frac{x}{m} + \frac{x}{m^3} - \frac{1}{m^2} - \frac{x}{m} - \frac{x}{m^3} - \frac{x}{m^5}\right)\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A132}$$

$$= \left(\frac{1}{m^2} + \frac{x}{m^5}\right) \quad \left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A133}$$

$$= \frac{1}{m^4}\ \left(1+\frac{x}{m^3}\right)^2 \left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^2}}\right)^2 \qquad \text{A134}$$

Considering the second part, it is seen that

$$\frac{1}{1+\frac{1}{m^2}}\left(1+\frac{1}{m^2}\left(\frac{x}{m}-\frac{1}{m^2}\ \frac{1}{1+\frac{1}{m^2}}\right)\left(\frac{1}{1+\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}}\right)\right)^2 \qquad \text{A135}$$

$$=\left(\frac{1}{1+\frac{1}{m^2}}\ 1+\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}+\frac{1}{m^2}\ \left(\frac{x}{m}-\frac{1}{m^2}\ \frac{1}{1+\frac{1}{m^2}}\right)\right)\left(\frac{1+\frac{1}{m^2}}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A136}$$

$$=\left(\frac{1}{1+\frac{1}{m^2}}\ 1+\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}+\frac{x}{m^3}-\frac{1}{m^4}\ \frac{1}{1+\frac{1}{m^2}}\right)^2\left(\frac{1+\frac{1}{m^2}}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A137}$$

$$=\frac{1}{1+\frac{1}{m^2}}\left(1+\frac{x}{m^3}\right)^2\frac{\left(1+\frac{1}{m^2}\right)^2}{\left(1+\frac{1}{m^2}+\frac{1}{m^4}\right)^2} \qquad \text{A138}$$

$$=\frac{1}{1+\frac{1}{m^2}}\ \left(1+\frac{x}{m^3}\right)^2\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A139}$$

Then the sum of these, or the complete exponential argument, is

$$\frac{1}{m^4}\left(1+\frac{x}{m^3}\right)\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2+\left(1+\frac{1}{m^2}\right)\left(1+\frac{x}{m^3}\right)^2\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A140}$$

$$= \left(1+\frac{1}{m^2}+\frac{1}{m^4}\right)\left(1+\frac{x}{m^3}\right)^2\left(\frac{1}{1+\frac{1}{m^2}+\frac{1}{m^4}}\right)^2 \qquad \text{A141}$$

$$= \left(1+\frac{x}{m^3}\right)^2 / 1+\frac{1}{m^2}+\frac{1}{m^4} \qquad \text{A142}$$

Thus the stationary phase point contribution simplifies to

$$-\sqrt{1+1/m^2}\ \sqrt{it/\pi}\ \frac{1}{2\sqrt{i\pi t}}\ \sqrt{\frac{\pi}{t\left(1+\left(\frac{1}{m^4}\right)\left(\frac{1}{1+\frac{1}{m^2}}\right)\right)}}$$

$$\cdot\left\{\frac{a_2 e^{\frac{-i\pi}{4}}\, e^{-it\left(\left(1-\frac{x}{m^3}\right)\ /1+\frac{1}{m^2}+\frac{1}{m^4}\right)}\cdot(1+1/m^2+1/m^4)}{\left(1-\frac{x}{m^3}\right)\ \left(1+\frac{1}{m^2}\right)}\right.$$

$$\left.+\ \frac{b_2 e^{\frac{-i\pi}{4}}\, e^{-it\left(1+\frac{x}{m^3}\right)/1+\frac{1}{m^2}+\frac{1}{m^4}}\ 1+1/m^2+1/m^4}{\left(1+\frac{x}{m^3}\right)\ \left(1+\frac{1}{m^2}\right)}\right\} \qquad \text{A143}$$

$$= -\sqrt{1+1/m^2}\ \sqrt{it/\pi}\ \ \frac{1}{2\sqrt{i\pi t}}\sqrt{\frac{1+1/m^2}{t\cdot 1+\frac{1}{m^2}+\frac{1}{m^4}}}\ \ \frac{1+1/m^2+1/m^4}{1+\frac{1}{m^2}}$$

$$\cdot\left\{\frac{a_2 e^{\frac{-i\pi}{4}}\, e^{-it\left(\left(1-\frac{x}{m^3}\right)^2/1+\frac{1}{m^2}+\frac{1}{m^4}\right)}}{1-\frac{x}{m^3}}\right.$$

$$\left.+\ \frac{b_2 e^{\frac{-i\pi}{4}}\, e^{-it\left(\left(1+\frac{x}{m^3}\right)^2/1+\frac{1}{m^2}+\frac{1}{m^4}\right)}}{1+\frac{x}{m^3}}\right\} \qquad \text{A144}$$

$$= \frac{-\sqrt{i}\ \sqrt{1+1/m^2+1/m^4}}{2\sqrt{i\pi t}} \left\{ \frac{a_2 e^{\frac{-i\pi}{4}} e^{-it((1-\frac{x}{m^3})\ /1+\frac{1}{m^2}+\frac{1}{m^4})}}{1-\frac{x}{m^3}} \right.$$

$$\left. + \frac{b_2 e^{\frac{-i\pi}{4}} e^{-it((1+\frac{x}{m^3})^2/1+\frac{1}{m^2}+\frac{1}{m^4})}}{1+\frac{x}{m^3}} \right\} \qquad \text{A145}$$

And this is seen to be, since $e^{\frac{-i\pi}{4}}\sqrt{i} =$,

$$a_2 F_3(x)+b_2 G_3(x) \qquad \text{A146}$$

Now, considering the end point contributions, the denominators are expanded and the 2's are factored out to get

$$\frac{-\sqrt{1+1/m^2}}{4\sqrt{i\pi t}}\quad it/\ \frac{i}{t}\left\{ \frac{a_2 e^{-it(1-\frac{x}{m})^2}\ e^{-\frac{it}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2}}{(1-\frac{1}{m^2})(1-\frac{x}{m}-\frac{1}{m^2+1}(1-\frac{1}{m^2}))} \right.$$

$$- \frac{a_2 e^{-it(1+\frac{x}{m})^2}\ e^{-\frac{it}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2}}{(1+\frac{1}{m^2})(-1-\frac{x}{m}-\frac{1}{m^2+1}(1+\frac{1}{m^2}))} + \frac{b_2 e^{-it(1-\frac{x}{m})^2}\ e^{-\frac{it}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2}}{(1+\frac{1}{m^2})(1-\frac{x}{m}+\frac{1}{m^2+1}(1+\frac{1}{m^2}))}$$

$$\left. + \frac{b_2 e^{-it(1+\frac{x}{m})^2}\ e^{-\frac{it}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2}}{(1-\frac{1}{m^2})(-1-\frac{x}{m}+\frac{1}{m^2+1}(1-\frac{1}{m^2}))} \right\} \qquad \text{A147}$$

Once more in the denominators, terms of $1/m$ or higher are

neglected, and it is seen that the end point contributions are

$$-\frac{1+\frac{1}{m^2}}{4\sqrt{i\pi t}}\sqrt{it/\pi}\;\frac{i}{t}\left\{\frac{a_2 e^{-it(1-\frac{x}{m})^2}\;e^{-\frac{it}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2}}{\left(1-\frac{1}{m^2}\right)\left(1-\frac{x}{m}\right)}\right.$$

$$+\frac{a_2 e^{-it(1+\frac{x}{m})^2}\;e^{-\frac{it}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2}}{\left(1+\frac{x}{m}\right)\left(1+\frac{1}{m^2}\right)}+\frac{b_2 e^{-it(1-\frac{x}{m})^2}\;e^{-\frac{it}{1+\frac{1}{m^2}}(1+\frac{1}{m^2})^2}}{\left(1-\frac{x}{m}\right)\left(1+\frac{1}{m^2}\right)}$$

$$\left.+\frac{b_2 e^{-it(1+\frac{x}{m})^2}\;e^{-\frac{it}{1+\frac{1}{m^2}}(1-\frac{1}{m^2})^2}}{\left(1+\frac{x}{m}\right)\left(1-\frac{1}{m^2}\right)}\right\} \qquad \text{A148}$$

The outer constant is seen to be

$$\frac{\sqrt{1+1/m^2}}{4i\pi t}=\frac{\sqrt{1+1/m^2}}{(-2\sqrt{i\pi t})^2} \qquad \text{A149}$$

and there the endpoint contribution is approximately

$$a_2F_1(x)F_2(1)+a_2G_1(x)F_2(-1)$$
$$-b_2F_1(x)G_2(1)+b_2G_1(x)G_2(-1) \qquad \text{A150}$$

Thus it is seen that, after adding,

$$I_1+I_2\simeq a_1F_2(x)+b_1G_2(x)+a_2F_3(x)+b_2G_3(x)$$
$$+F_1(x)(a_1F_1(1)+b_1G_1(1)+a_2F_2(1)+b_2G_2(1))$$
$$+G_1(x)(a_1F_1(-1)+b_1G_1(-1)+a_2F_2(-1)+b_2G_2(-1))$$
$$\text{A151}$$

From this it is concluded that

$$\sqrt{it/\pi}\sum_{n=1}^{N}\int_{-1}^{1} e^{-it(y-\frac{x}{m})^2} a_nF_n(y)+b_nG_n(y))\ dy$$

$$= \sum_{n=1}^{N}\{a_nF_{n+1}(x) + b_nG_{n+1}(x)\}$$

$$+F_1(x)\sum_{n=1}^{N}\{a_nF_n(1)+b_nG_n(1)\}$$

$$+G_1(x)\sum_{n=1}^{N}\{a_nF_n(-1)+b_nG_n(-1)\} \qquad \text{A152}$$

And in turn, adding the I_0 term, it is had that

$$\sqrt{it/\pi}\int_{-1}^{1} e^{-it(y-\frac{x}{m})} (1+\Sigma\{a_nF_n(y)+b_n(F_n(y)\})\,dy$$

$$= 1 + F_1(x) + G_1(x) + \sum_{n=1}^{N}\{a_nF_{n+1}(x) + b_nG_{n+1}(x)\}$$

$$+ G_1(x)\sum_{n=1}^{N}\{a_nF_n(-1) + b_nG_n(-1)\} \qquad \text{A153}$$

Appendix B

This appendix simplifies the definite integral produced by assuming in 2.1.16 that a_2 is effectively infinite.

$$\frac{i}{\lambda L}\int_{-\infty}^{\infty} e^{-\frac{ik}{2L}\left[g_1 x_1'^2+g_2 x_2'^2-2x_1'x_2'\right]} e^{-\frac{ik}{2L}\left[g_1 x_1'^2+g_2 x_2^2-2x_1'x_2\right]} dx_1'$$

$$= \frac{1}{\lambda L}\int_{-\infty}^{\infty} e^{-\frac{ik}{2L}\left[g_1 x_1'^2+g_2 x_2'^2-2x_1'x_2' g_1 x_1'^2+g_2 x_2^2-2x_1'x_2\right]} dx_1' \qquad \text{B1}$$

$$= \frac{i}{\lambda L}\int_{-\infty}^{\infty} e^{-\frac{ik}{2L}\left[2g_1 x_1'^2-2(x_2'+x_2)x_1'+g_2(x_2'^2+x_2')\right]} dx_1' \qquad \text{B2}$$

$$= \frac{i}{\lambda L}\, e^{-\frac{ik}{2L}\, g_2(x_2'^2+x_2^2)} \int_{-\infty}^{\infty} e^{-\frac{ik}{L}\left[g_1 x_1'^2-(x_2'+x_2)x_1'\right]} dx_1' \qquad \text{B3}$$

The square is completed in the exponent by adding and subtracting $b^2/4a$, or

$$\frac{i}{\lambda L}\, e^{-\frac{ik}{2L}g_2(x_2'^2+x_2^2)} \int_{-\infty}^{\infty} e^{-\frac{ik}{L}\left[g_1 x_1'^2-(x_2+x_2)x_1'+\frac{(x_2'+x_2)}{4g_1}\right]}$$

$$e^{\frac{ik}{L}\frac{(x_2'+x_2)}{4g_1}} dx_1' \qquad \text{B4}$$

$$= \frac{i}{\lambda L}\, e^{-\frac{ik}{2L} g_2 (x_2'^2 + x_2^2)} \int_{-\infty}^{\infty} e^{-\frac{ik}{L}\left[\sqrt{g_1}\, x_1 - \frac{x_2' + x_2}{2\sqrt{g_1}}\right]^2} e^{\frac{ik}{L} \frac{(x_2' + x_2)^2}{4 g_1}} dx_1' \qquad \text{B5}$$

$$= \frac{i}{\lambda L}\, e^{-\frac{ik}{2L} g_2 (x_2' + x_2^2)} e^{\frac{ik}{2L} \frac{(x_2' + x_2)^2}{2g}} \int_{-\infty}^{\infty} e^{-\frac{ik}{L}\left[\sqrt{g_1}\, x_1' - \frac{x_2' + x_2}{2\sqrt{g_1}}\right]^2} dx_1' \qquad \text{B6}$$

If

$$\beta = \frac{-K}{L} \qquad \text{B7}$$

and

$$V = \sqrt{g_1}\, x_1' - \frac{x_2' + x_2}{2\sqrt{g_1}} \qquad \text{B8}$$

then the result is

$$dV = \sqrt{g_1}\; dx_1' \qquad \text{B9}$$

$$= \frac{i}{\lambda L}\, e^{-\frac{ik}{2L}\left[g_2 (x_2'^2 + x_2^2)\right]} e^{\frac{ik}{2L} \frac{(x_2' + x_2)^2}{2 g_1}} \int_{-\infty}^{\infty} \frac{e^{-i\beta V^2}}{\sqrt{g_1}}\, dV \qquad \text{B10}$$

Similarly letting

$$\sqrt{\beta} V = w \qquad \text{B11}$$

$$dw = \sqrt{\beta} dV \qquad \text{B12}$$

the result is

$$= \frac{i}{\lambda L}\, e^{-\frac{ik}{2L} g_2 (x_2'^2 + x_2^2)} e^{\frac{ik}{2L} \frac{(x_2' + x_2)^2}{2 g_1}} \frac{1}{\sqrt{g_1 \beta}} \frac{1+i}{\sqrt{\pi/2}} \qquad \text{B13}$$

then

$$= \frac{i}{\lambda L}\,(1+i)\sqrt{\pi/2}\,\sqrt{L/-g_1K}\; e^{-\frac{ik}{2L}\left[g_2 x_2'^2 + g_2 x_2^2 - \frac{x_2'^2}{2g_1} - \frac{2x_2x_2'}{2g_1} - \frac{x_2^2}{2g_1}\right]} \qquad \text{B14}$$

$$= \frac{1+i}{\sqrt{2}}\,\frac{i}{\lambda L}\,\sqrt{\pi}\sqrt{\frac{\lambda L}{2g_1\pi}}\; e^{-\frac{ik}{2L}\left[\frac{2g_1g_2x_2'^2 + g_1g_2x_2^2 - x_2'^2 - 2x_2x_2' - x_2^2}{2g_1}\right]} \qquad \text{B15}$$

$$= \sqrt{i}\,\sqrt{\frac{1}{2L\lambda g_1}}\; e^{-\frac{ik}{2L\cdot 2g_1}\left[(2g_1g_2-1)(x_2'^2+x_2^2) - 2x_2x_2'\right]} \qquad \text{B16}$$

which is the final kernel.

Appendix C

This appendix simplifies the integral equation in 2.1.29 into the final form.

$$\gamma u(x) = \int_{-1}^{1} \sqrt{iF}\,u(y)e^{-i\pi F[g(x^2+y^2)-2xy]} \qquad \text{C1}$$

Let

$$u(x) = g(x)e^{-\frac{i}{2m}(m-1)x} \qquad \text{C2}$$

since

$$N = \frac{m-1}{2m} F \qquad \text{C3}$$

then

$$\gamma g(x)e^{-i\pi F\frac{m^2-1}{2m}x^2} = \sqrt{iF}\int_{-1}^{1} g(y)e^{-i\pi F\frac{m^2-1}{2m}y^2}\, e^{-i\pi F[g(x^2+y^2)-2xy]}\, dy \qquad \text{C4}$$

However, since

$$m = \frac{\sqrt{g+1} + \sqrt{g-1}}{\sqrt{g+1} - \sqrt{g-1}} \qquad \text{C5}$$

$$= \frac{(\sqrt{g+1} + \sqrt{g-1})^2}{g+1-g+1} \qquad \text{C6}$$

$$= \frac{2g + 2\sqrt{(g+1)(g-1)}}{2} \qquad C7$$

$$= g + \sqrt{g^2-1} \qquad C8$$

so

$$m^2 = g^2+2g\sqrt{g^2-1}+g^2-1 \qquad C9$$

$$m^2+1 = 2g^2+2g\sqrt{g^2-1} \qquad C10$$

$$\frac{m^2+1}{2} = g^2+g\sqrt{g^2-1} \qquad C11$$

If both sides are divided by m, or rather one by m and the other by its equivalent, $g+\sqrt{g^2-1}$, the result is

$$\frac{m^2+1}{2m} = g \qquad C12$$

this can then be substituted into the integral in place of g:

$$\gamma g(x) = \sqrt{iF}\int_{-1}^{1} g(y)e^{-i\pi F\left[\frac{m^2+1}{2m}x^2+\frac{m^2+1}{2m}y^2-2xy+\frac{m^2-1}{2m}y^2-\frac{m^2-1}{2m}x^2\right]}dy \qquad C13$$

$$= \sqrt{iF}\int_{-1}^{1} g(y)e^{-i\pi F\left[\frac{x^2}{m}+my^2-2xy\right]}dy \qquad C14$$

$$= \sqrt{iF}\int_{-1}^{1} g(y)e^{-i\pi F(y-\frac{x}{m})^2}dy \qquad C15$$

Appendix D

List of program BARC employing the expressions developed in Chapter III and IV.

```
      PROGRAM BARC(INPUT,OUTPUT,TAPE8=OUTPUT)
      REAL NEQ,MAG,MSUBN(51),MSUPN(51),INTENS(1000)
      COMPLEX EYE,COEF(51),AN(51),LAMBDA(51)
      COMPLEX CL(51),CONST(51),CDUM,AN1,AN2,RTEYE
      COMPLEX FIELDX(1000),SIG,BN1,BN2,FNX,ROOT
      DIMENSION LABEL(17),STOREX(1000),NINDEX(51),PLOCON(51)
      DIMENSION FUNVAL(20,410),PLOFUN(20)
      DATA LABEL/17(10H          )/
C
C
C  THIS PROGRAM COMPUTES RESONATOR MODE EIGENVALUES AND
C   SUBSEQUENTL  EVALUATES INTENSITY VALUES FOR POINTS
C   ACROSS THE OUTPUT PLANE OF A STRIP LASER RESONATOR.
C   THE PROGRAM DEALS WITH EITHER A BARE OR LOADED
C   CAVITY, USER'S PREFERENCE.
C  OUTPUT CONSISTS OF AN EIGENVALUE LIST, WITH PHASE
C   AND MAGNITUDE, FIELD VALUES FOR A SELECTED MODE
C   EITHER ON OR OFF THE MIRROR, PLOTS OF FIELD SERIES
C   FUNCTIONS OR WEIGHTING CONSTANTS, AND PLOTS OF INTENSITY
C   ACROSS THE OUTPUT PLANE WITH EITHER LIMITED OR EXTENTED
C   RANGE.
C  COMPILED CODE NEEDED AROUND 110000 OCTAL TO LOAD.
C
C  INPUT QUANTITIES ARE AS FOLLOWS:
C
C   MAG = CAVITY MAGNIFICATION
C   NEQ = EQUIVALENT FRESNEL NUMBER
C   MTEST1 = FIELD SOLUTION PARITY DESIGNATOR
C   NBIG = DESIRED # TERMS IN FIELD SERIES
C   CAVLEN = CAVITY LENGTH IN LENGTH UNITS FOR LOADED CASE
C   GNAWT = SMALL SIGNAL GAIN IN PER LENGTH
C   H = AVERAGE CAVITY INTENSITY, OR EIGENVALUE FORCING
C      PARAMETER
C
C
C   TO TERMINATE PROGRAM, INPUT MAG=0 OR LESS.
C   NOTE:  EVMAG DENOTES EIGENVALUE,MAGNITUDE AND EVPH
C  DENOTES EIGENVALUE, PHASE.
C
C  THIS PROGRAM ALSO REQUIRES IMSL ROUTINE ZCPOLY AND PLOT
C   LIBRARY CCPLOT56X. FINAL COPY, 20 OCT 1980. J E ROWLEY
C
C
C
994   FORMAT(G10.3)
      LABEL(1)=10H   NEQ=
      LABEL(3)=10H   MAG=
777   WRITE(8,999)
999   FORMAT(1H1,1X,*INPUT MAG, NEQ, AND PARITY: *,/)
      READ *,MAG,NEQ,MTEST1
      IF(MAG.LE.0.) GO TO 888
```

```
      WRITE(8,88)MAG,NEQ
      IF(MTEST1.EQ.0) GO TO 8
      WRITE(8,977)
      GO TO 9
8     WRITE(8,976)
9     CONTINUE
977   FORMAT(1X,*PARITY IS ODD. *,/)
976   FORMAT(1X,*PARITY IS EVEN. *,/)
      LABEL(5)=10H    MODE E
      LABEL(6)=10HIGENVALUE:
C
C   MSUPN(I)=MAG**(I-1)
C   MSUBN(I)=1+1/MAG**2 + ... +1/MAG**(2*I-2)
C
      MSUBN(1)=1.0
      MSUPN(1)=1.0
      DO 10 I=2,51
      MSUPN(I)=MAG*MSUPN(I-1)
      MSUBN(I)=MSUBN(I-1)+1./MSUPN(I)**2
10    CONTINUE
C
      PI=2.*ASIN(1.0)
      EYE=CMPLX(0.,1.)
      RTEYE=CMPLX(1.,1.)/SQRT(2.)
      DUM1=2*PI*MAG**2/(MAG**2-1.)
      RNBIG=ALOG(250*NEQ)/ALOG(MAG)
      IF(RNBIG.LE.50.) GO TO 15
      WRITE(8,998)
      GO TO 777
15    WRITE(8,996)RNBIG
996   FORMAT(1X,*CALCULATED NBIG = *,G14.7,*INPUT INTEGER CHOICE:*,/)
      READ *,NBIG
      WRITE(8,979)NBIG
      WRITE(8,975)
975   FORMAT(1X,*TYPE 1 FOR GAIN CONSIDERATION : *,/)
      READ *,IGAINQ
      WRITE(8,979)IGAINQ
      IF(IGAINQ.NE.1) GO TO 5
      WRITE(8,974)
974   FORMAT(1X,*INPUT LENGTH AND S-S-GAIN IN COMMON UNITS : *,/)
      READ *,CAVLEN,GNAWT
      WRITE(8,971)CAVLEN,GNAWT
C
C   DIVIDE INPUT INTENSITY GAIN BY TWO TO MAKE IT THE FIELD GAIN,
C   WHICH IS WHAT THIS PROGRAM ACTUALLY REQUIRES
C
C
      GNAWT=GNAWT/2.
971   FORMAT(1X,*INPUT VALUES ARE : *,2G14.7,/)
      H=SQRT(GNAWT*CAVLEN/ALOG(MAG)-.5)
      WRITE(8,973)H
```

```
973   FORMAT(1X,*H=*,G14.7,*INPUT MODIFIED VALUE OR 0 TO CONT : *,/)
      READ *,HVAL
      WRITE(8,972)HVAL
972   FORMAT(1X,*INPUT VALUE IS : *,G14.7,/)
      IF(HVAL.NE.0.) H=HVAL
      GAMMA=EXP(2*CAVLEN*(GNAWT/(1.+2*H**2)))
      GO TO 6
5     H=1. $ GAMMA=1.
6     CONTINUE
      WRITE(8,993)
993   FORMAT(1X,*INPUT ZERO TO LIST EIGENVALUES :*,/)
      READ *,LTEST
      WRITE(8,979)LTEST
      LABEL(13)=10H     EVEN
      NDEG=NBIG+1
      IF(MTEST1.EQ.0) GO TO 16
      LABEL(13)=10H      ODD
      NDEG=NBIG
16    T=DUM1*NEQ
      LABEL(14)=10H PARITY
C
C   COMPUTE COEFFICIENTS OF THE POLYNOMIAL
C   P(Z)=COEF(1)*Z**NDEG + COEF(2)*Z**(NDEG-1) + ... +
C   COEF(NDEG)*Z + COEF(NDEG+1)
C
      COEF(1)=CMPLX(1.,0.)
      NCOEF=NDEG+1
      DO 25 I=1,NDEG
      AN1=RTEYE*2*SQRT(PI*T/MSUBN(I))
      AN2=-T*EYE/MSUBN(I)
      AN3=1.-1./MSUPN(I+1)
      AN4=1.+1./MSUPN(I+1)
      AN(I)=(CEXP(AN2*AN3**2)/AN3+CEXP(AN2*AN4**2)/AN4)/AN1
      IF(MTEST1.EQ.0) GO TO 25
      AN(I)=(CEXP(AN2*AN3**2)/AN3-CEXP(AN2*AN4**2)/AN4)/AN1
25    CONTINUE
      IF(MTEST1.EQ.1) GO TO 27
      COEF(2)=(AN(1)-1.)*GAMMA
      DO 26 I=3,NCOEF
26    COEF(I)=GAMMA**(I-1)*(AN(I-1)-AN(I-2))
      GO TO 666
27    DO 28 I=2,NCOEF
28    COEF(I)=AN(I-1)*GAMMA**(I-1)
C
C   COMPUTE ROOTS OF POLYNOMIAL WITH IMSL ROUTINE TO
C   OBTAIN THE EIGENVALUES
C
666   CALL ZCPOLY(COEF,NDEG,LAMBDA,IER)
C
C   NOW ORDER THE EIGENVALUES BY SIZE
C
```

```
      IF(LTEST.EQ.0) WRITE(8,89)
      I=1
      DO 70 I1=2,NDEG
      SIZE= REAL(LAMBDA(I))**2+AIMAG(LAMBDA(I))**2
      K=I
      DO 75 J=I1,NDEG
      SIZE1=REAL(LAMBDA(J))**2+AIMAG(LAMBDA(J))**2
      IF(SIZE1.LT.SIZE) GO TO 75
      K=J
      SIZE=SIZE1
75    CONTINUE
      CDUM=LAMBDA(I)
      LAMBDA(I)=LAMBDA(K)
      LAMBDA(K)=CDUM
      CL(I)=LAMBDA(I)
      EVPH=ATAN2(AIMAG(CL(I)),REAL(CL(I)))*180./PI
      SMA=REAL(CL(I))**2+AIMAG(CL(I))**2
      SMAG=SQRT(SMA)
      IF(LTEST.EQ.0) WRITE(8,333)I,LAMBDA(I),SMAG,EVPH
333   FORMAT(1X,I10,4(G14.7,1X),/)
      I=I1
70    CONTINUE
      EVPH=ATAN2(AIMAG(LAMBDA(NDEG)),REAL(LAMBDA(NDEG)))*180./PI
      SMA=REAL(LAMBDA(NDEG))**2+AIMAG(LAMBDA(NDEG))**2
      SMAG=SQRT(SMA)
      IF(LTEST.EQ.0) WRITE(8,333)NDEG,LAMBDA(NDEG),SMAG,EVPH
C
C    NOW CALCULATE THE CONSTANTS FOR THE FUNCTION SUM FOR A
C   PRTICULAR MODE. LOOP THEN TO CALCULATE THE FIELD AT A SELECTED
C   NUMBER OF POINTS FROM ZERO TO ONE
C
45    X=0.
      BRIGHT=0.
      WRITE(8,997)
997   FORMAT(1X,*INPUT 1 TO CALC FIELDS, 0 TO DO NEW CAVITY:*,/)
      READ *,MTEST2
      WRITE(8,979)MTEST2
      IF(MTEST2.EQ.0) GO TO 777
      WRITE(8,995)
995   FORMAT(1X,*INPUT DESIRED MODE NUMBER:*,/)
      READ *,MODE
      WRITE(8,979)MODE
      LABEL(15)=10H    MODE #
      ENCODE(10,987,LABEL(16))MODE
987   FORMAT(I2,8X)
      NPOINT=0
      NL=NBIG
      ROOT=LAMBDA(MODE)
992   FORMAT(1X,I5,1X,*POINTS WILL BE EVALUATED FOR MODE *,2G14.7,/)
C
C     CONSIDER I/O OPTIONS AND CALCULATE THE CONSTANTS FOR EITHER
```

```
C   PARITY CHOICE.
C
      IF(MTEST1.EQ.1) GO TO 40
      DO 30 I=1,NL
      RINDEX(I)=I
30    CONST(I)=H*(ROOT-GAMMA)/GAMMA/AN(NDEG)*(ROOT/GAMMA)**(NL-I)
      GO TO 29
40    DO 41 I=1,NL
      RINDEX(I)=I
41    CONST(I)=(GAMMA/ROOT)**I
29    WRITE(8,982)
982   FORMAT(1X,*INPUT 0 TO CALC INTENSITIES OVER EXPANDED RANGE:*,/)
      READ *,JTEST
      WRITE(8,979)JTEST
      ENCODE(10,994,LABEL(2))NEQ
      ENCODE(10,994,LABEL(4))MAG
      ENCODE(10,994,LABEL(7))REAL(ROOT)
      ENCODE(10,994,LABEL(8))AIMAG(ROOT)
      IF(JTEST.NE.0) GO TO 46
      LABEL(9)=10HMIRROR
      LABEL(10)=10HPLANE
      LABEL(11)=10HSCALED
      IF(IGAINQ.EQ.1) LABEL(11)=10H
      LABEL(12)=10H INTENSITY
      LABEL(17)=10HAPPROX # 2
      CALL ALLINT(MAG,MSUBN,MSUPN,CONST,T,NBIG,MTEST1,ROOT,LABEL,H,
     1   GAMMA,IGAINQ)
      WRITE(8,981)
981   FORMAT(1X,*INPUT 0 TO CONTINUE WITH OTHER I/O OPTIONS:*,/)
      READ *,JTEST1
      WRITE(8,979)JTEST1
      IF(JTEST1.NE.0) GO TO 45
46    LABEL(17)=10HAPPROX # 1
      WRITE(8,980)
980   FORMAT(1X,*INPUT # PNTS FROM 0-1 AND 0-1 TO PLOT OR PRINT:*,/)
      READ *,INCX,MTEST3
      WRITE(8,978)INCX,MTEST3
      WRITE(8,992)
C
C    CALCULATE THE FIELD AT VARIOUS X VALUES USING THE
C   CONSTANTS JUST CALCULATED:
C   FOR EVEN, F(X)= 1 + SUM(H(X))
C   FOR ODD, F(X)= SUM(H(X)),  WHERE THE H(X)'S ARE THE ONES
C   DERIVED FOR EACH CASE
31    NPOINT=NPOINT+1
      STOREX(NPOINT)=X
      SIG=CMPLX(0.,0.)
      DO 32 I=1,NL
      BN1=RTEYE*2*SQRT(PI*T/MSUBN(I))
      BN2=-T*EYE/MSUBN(I)
      BN3=1.-X/MSUPN(I+1)
```

```
      BN4=1.+X/MSUPN(I+1)
      HNX=(CEXP(BN2*BN3**2)/BN3+CEXP(BN2*BN4**2)/BN4)/BN1
      FUNVAL(I,NPOINT)=REAL(-HNX)**2+AIMAG(-HNX)**2
      IF(MTEST1.EQ.0) GO TO 32
      HNX=(CEXP(BN2*BN3**2)/BN3-CEXP(BN2*BN4**2)/BN4)/BN1
      FUNVAL(I,NPOINT)=REAL(-HNX)**2+AIMAG(-HNX)**2
32    SIG=SIG+CONST(I)*HNX
      IF(MTEST1.EQ.1) GO TO 33
      FIELDX(NPOINT)=H+SIG
      GO TO 34
33    FIELDX(NPOINT)=SIG
34    INTENS(NPOINT)=REAL(FIELDX(NPOINT))**2+AIMAG(FIELDX(NPOINT))**2
      IF(INTENS(NPOINT).GT.BRIGHT) BRIGHT=INTENS(NPOINT)
      IF(MTEST3.EQ.0) GO TO 35
      WRITE(8,87)X,FIELDX(NPOINT)
      WRITE(8,86)INTENS(NPOINT)
86    FORMAT(1X,*INTENSITY = *,G14.7,/)
87    FORMAT(1X,*X = *,G14.7,* FIELD = *,2G14.7)
35    X=X+1./INCX
      IF(NPOINT.LT.INCX) GO TO 31
      IF(MTEST3.EQ.1) GO TO 777
      WRITE(8,991)
991   FORMAT(1X,*TYPE ZERO TO PLOT CONSTANTS VS N:*,/)
      READ *,MTESTC
      WRITE(8,979)MTESTC
      IF(MTESTC.NE.0) GO TO 38
      LABEL(9)=10HCONSTANT #
      LABEL(10)=10H
      LABEL(11)=10H  MOD(CONS
      LABEL(12)=10HTANT)**2
      DO 42 I=1,NL
42    PLOCON(I)=REAL(CONST(I))**2+AIMAG(CONST(I))**2
      CALL HGRAPH(RINDEX,PLOCON,NBIG,LABEL,1,-1,11)
      WRITE(8,984)MODE
984   FORMAT(1X,*COMPLETED PLOT OF CONSTANTS, MODE =*,I2,/)
38    WRITE(8,990)
990   FORMAT(1X,*TYPE INDEX OF FUNCTION TO PLOT OR 0 TO CONTINUE:*,/)
      READ *,INDEX
      WRITE(8,979)INDEX
      LABEL(9)=10HMIRROR
      LABEL(10)=10HPLANE
      IF(INDEX.EQ.0) GO TO 36
      DO 43 I=1,INCX
43    PLOFUN(I)=FUNVAL(INDEX,I)
      LABEL(11)=10HMOD(FUN,IN
      ENCODE(10,989,LABEL(12))INDEX
989   FORMAT("DEX=",I2,")**2")
      CALL HGRAFH(STOREX,PLOFUN,INCX,LABEL,1,0,0)
      WRITE(8,986)INDEX
986   FORMAT(1X,*COMPLETED PLOT OF FUNCTION, INDEX =*,I2,/)
      GO TO 38
```

```
36    WRITE(8,988)
988   FORMAT(1X,*TYPE ZERO TO PLOT INTENSITY:*,/)
      READ *,ICONT
      WRITE(8,979)ICONT
      IF(ICONT.NE.0) GO TO 45
      LABEL(11)=10HSCALED
      IF(IGAINQ.EQ.1) LABEL(11)=10H
      LABEL(12)=10H INTENSITY
      IF(IGAINQ.EQ.1) BRIGHT=1.
      DO 37 I=1,INCX
37    INTENS(I)=INTENS(I)/BRIGHT
      CALL HGRAPH(STOREX,INTENS,INCX,LABEL,1,0,0)
      WRITE(8,985)
      GO TO 45
978   FORMAT(1X,*INPUT VALUES ARE : *,2I5,/)
979   FORMAT(1X,*INPUT VALUE IS : *,I5,/)
985   FORMAT(1X,*COMPLETED PLOT OF NORMALIZED INTENSITY.*,/)
998   FORMAT(1X,*REVISE PARAMETERS SO NDEG  50*)
88    FORMAT(10X,*MAG = *,F6.2,5X,*NEQ = *,F6.2,/)
89    FORMAT(9X,*I*,2X,*LAMBDA(REAL)*,2X,*LAMBDA(IMAG)*,6X,
     1  *EVMAG*,11X,*EVPH*,/)
888   CALL EXIT
      END
      SUBROUTINE ALLINT(MAG,MSUBN,MSUPN,CONST,T,NBIG,MTEST1,ROOT
     1    ,LABEL,H,GAMMA,IGAINQ)
      DIMENSION LABEL(17),XSAVE(2000)
      REAL MSUBN(51),MSUPN(51),INARG1,INARG2,INARG3,INARG4,INARG5
      REAL INARG6,INARG7,INARG8,MAG,INTENS(2000)
      REAL MINV
      COMPLEX APART1,APART2,BPART1,BPART2,ALLFUN,CONST(51),ROOT,EYE
      COMPLEX AFUN,BFUN,SPNTC,SPNTD,EVENX,OUTCON,FRESL,CONSTA,CONSTB
      COMPLEX EYEFAC,SPCON
C
C
C    THIS SUBROUTINE FOLLOWS PROGRAM BARC AND COMPUTES BEAM INTENSITITE
C    IN THE OUTPUT PLANE FROM THE OPTIC AXIS TO SOME DESIRED OUTER POIN
C    OUTER POINT AND # INTERMEDIATE POINTS FOR EVALUATION ARE INPUT
C    WHILE ALL OTHER REQUIRED QUANTITIES ARE CARRIED THROUGH IN THE
C    ARGUMENT LIST AS FOLLOWS:
C
C    MAG= CAVITY MAGNIFICATION
C    MSUBN= ARRAY FOR PARTIAL SUMS OF INVERSE POWERS OF MAG
C    MSUPN= ARRAY FOR MAG TO SOME POWER
C    CONST= ARRAY OF CONSTANTS IN THE ASYMPTOTIC SERIES
C    T= QUANTITY DEFINED IN BARC PER HORWITZ
C    NBIG= # TERMS IN THE SERIES
C    MTEST1= PARITY DESIGNATOR
C    ROOT= MODE EIGENVALUE
C    LABEL= PLOT LABELING ARRAY
C
C
```

```
      BRIGHT=0.
      PI=2.*ASIN(1.)
      EYE=CMPLX(0.,1.)
      EYEFAC=(1.-EYE)/2.
      WRITE(8,900)
900   FORMAT(1H1,*ENTERING EXTENDED RANGE INTENSITY SUBROUTINE.*,/)
      DO 10 I=1,51
10    MSUPN(I)=MAG*MSUPN(I)
      NPOINT=1
      X=0.
      WRITE(8,901)
901   FORMAT(1X,*INPUT MIN AND MAX X VALUES AND # POINTS BETWEEN: *,/)
      READ *,XMIN,XMAX,INCX
      X=XMIN
50    ALLFUN=(0.,0.)
      XOMAG=X/MAG
      DO 310 I=1,NBIG
      MINV=1./MSUPN(I)
      EVENX=(0.,0.)
      SPNTD=(0.,0.)
      SPNTC=(0.,0.)
      CONSTA=-CONST(I)*GAMMA
      CONSTB=CONSTA
      IF(MTEST1.EQ.1) CONSTB=-CONSTA
      P2PRYM=2.*(1.+1./MSUPN(2*I)/MSUBN(I))
      STAPHA=(MINV/MSUBN(I)+XOMAG)/(.5*P2PRYM)
      STAPHB=(-MINV/MSUBN(I)+XOMAG)/(.5*P2PRYM)
      INARG1=(1.-XOMAG)**2+((1.-MINV)**2/MSUBN(I))
      INARG2=((1.-XOMAG-(MINV-MINV**2)/MSUBN(I))**2)/(.5*P2PRYM)
      AARG1=INARG2-INARG1
      APART1=CEXP(EYE*T*AARG1)/(1.-MINV)*(-CONSTA)
      INARG3=(-1.-XOMAG)**2+(1.+MINV)**2/MSUBN(I)
      INARG4=((-1.-XOMAG-(MINV+MINV**2)/(MSUBN(I)))**2)/(.5*P2PRYM)
      AARG2=INARG4-INARG3
      APART2=CEXP(EYE*T*AARG2)*(-CONSTA)/(1.+MINV)
      INARG5=(1.-XOMAG)**2+(1.+MINV)**2/MSUBN(I)
      INARG6=(1.-XOMAG+(MINV+MINV**2)/MSUBN(I))**2/(.5*P2PRYM)
      BARG1=INARG6-INARG5
      BPART1=CEXP(BARG1*EYE*T)*(-CONSTB)/(1.+MINV)
      INARG7=(1.+XOMAG)**2+(1.-MINV)**2/MSUBN(I)
      INARG8=(-1.-XOMAG+(MINV-MINV**2)/MSUBN(I))**2/(.5*P2PRYM)
      BARG2=INARG8-INARG7
      BPART2=CEXP(BARG2*EYE*T)*(-CONSTB)/(1.-MINV)
      OUTCON=SQRT(MSUBN(I)/PI/T/P2PRYM)/2./ROOT
      SPCON=SQRT(MSUBN(I)*2./P2PRYM/PI/T)/2./ROOT
      FRSPOS=SQRT(T/PI/P2PRYM)*(2.*(1.-XOMAG)-2.*(1.-MINV)/MSUPN(I)/
     1   MSUBN(I))
      FRSNEG=SQRT(T/PI/P2PRYM)*(2.*(-1.-XOMAG)-2.*(1.+MINV)/MSUPN(I)/
     1   MSUBN(I))
      IF(STAPHA.GE.-1..AND.STAPHA.LE.1.) GO TO 130
      AFUN=APART1*(FRESL(FRSPOS)-EYEFAC)-APART2*(FRESL(FRSNEG)-EYEFAC)
```

```
      IF(STAPHA.LT.1.) GO TO 200
      AFUN=APART1*(FRESL(FRSPOS)+EYEFAC)-APART2*(FRESL(FRSNEG)+EYEFAC)
      GO TO 200
130   AFUN=APART1*(FRESL(FRSPOS)-EYEFAC)-APART2*(FRESL(FRSNEG)+EYEFAC)
      SPNTC=SPCON*(-CONSTA)/(1.-STAPHA/MSUPN(I))*CEXP(-EYE*(PI/4.+T*
     1   ((STAPHA-XOMAG)**2+(1.-STAPHA/MSUPN(I))**2/MSUBN(I))))
200   FRSPOS=SQRT(T/PI/P2PRYM)*(2.*(1.-XOMAG)+2.*(1.+MINV)/MSUPN(I)/
     1   MSUBN(I))
      FRSNEG=SQRT(T/PI/P2PRYM)*(2.*(-1.-XOMAG)+2.*(1.-MINV)/MSUPN(I)/
     1   MSUBN(I))
      IF(STAPHB.GE.-1..AND.STAPHB.LE.1.) GO TO 140
      BFUN=BPART1*(FRESL(FRSPOS)-EYEFAC)-BPART2*(FRESL(FRSNEG)-EYEFAC)
      IF(STAPHB.LT.1.) GO TO 300
      BFUN=BPART1*(FRESL(FRSPOS)+EYEFAC)-BPART2*(FRESL(FRSNEG)+EYEFAC)
      GO TO 300
140   BFUN=BPART1*(FRESL(FRSPOS)-EYEFAC)-BPART2*(FRESL(FRSNEG)+EYEFAC)
      SPNTD=SPCON*(-CONSTB)/(1.+STAPHB/MSUPN(I))*CEXP(-EYE*(PI/4.+T*
     1   ((STAPHB-XOMAG)**2+(1.+STAPHB/MSUPN(I))**2/MSUBN(I))))
300   ALLFUN=OUTCON*(AFUN+BFUN)+SPNTC+SPNTD+ALLFUN
310   CONTINUE
802   FORMAT(1X,2G14.7)
      EARG1=SQRT(T/2./PI)*2.*(1.-XOMAG)
      EARG2=SQRT(T/2./PI)*2.*(-1.-XOMAG)
      EVENX=CSQRT(EYE/2.)/ROOT*(FRESL(EARG1)-FRESL(EARG2))
      IF(X/MAG.GE.-1..AND.X/MAG.LE.1.) EVENX=EVENX-CSQRT(EYE/2.)/ROOT*
     1   (1.-EYE)+CEXP(-EYE*PI/4.)/ROOT*CSQRT(EYE)
      EVENX=EVENX*H*GAMMA
      IF(MTEST1.EQ.0) ALLFUN=ALLFUN+EVENX
      WRITE(8,802)ALLFUN
      INTENS(NPOINT)=AIMAG(ALLFUN)**2+REAL(ALLFUN)**2
      XSAVE(NPOINT)=X
      WRITE(8,800)INTENS(NPOINT),XSAVE(NPOINT)
800   FORMAT(1X,2G14.7)
      IF(INTENS(NPOINT).GT.BRIGHT) BRIGHT=INTENS(NPOINT)
      IF(IGAINQ.EQ.1) BRIGHT=1.
      IF(X.GT.XMAX) GO TO 500
      X=X+1./INCX
      NPOINT=NPOINT+1
      GO TO 50
500   DO 510 I=1,NPOINT
510   INTENS(I)=INTENS(I)/BRIGHT
      CALL HGRAPH(XSAVE,INTENS,NPOINT,LABEL,1,0,0)
      DO 600 I=1,51
600   MSUPN(I)=MSUPN(I)/MAG
      WRITE(8,904)
904   FORMAT(1X,*COMPLETED CALCULATION AND PLOT, EXTENDED.*,/)
      RETURN
      END
      SUBROUTINE HGRAPH(X,Y,N,ID,NO,NP,NS)
      DIMENSION X(1),Y(1),ID(1) $ IF (NO.EQ.2) CALL PLOT(-1.85,2.10,-3
      IF (NO.EQ.2) GO TO 30     $ IF (NO.LT.0) GO TO 10
```

```
      CALL SCALE(X,7.,N,1)       $ CALL SCALE(Y,5.,N,1)
10    CALL PLOT(0.,11.,2)  $  CALL PLOT(8.5,11.,2)
      CALL PLOT(8.5,0.,2) $  CALL PLOT(0.,0.,2)
      CALL PLOT(1.35,1.35,-3)  $  CALL PLOT(0.,8.30,-2)
      IF(ID(1).EQ.999) GO TO 25
      CALL PLOT(.1,-.1,-3)       $ CALL PLOT(0.,-2.,-2)
      DO 20 I=1,7,2
20    CALL SYMBOL( (I+1.5)*.1,.3,.07,ID(I),90.,20)
      CALL PLOT(0.,0.,3)         $ CALL PLOT(1.,0.,2)
      CALL PLOT(1.,2.,2)         $ CALL PLOT(0.,2.,-2)
      CALL PLOT(-.1,.1,-3)
25    CALL PLOT(5.8,0.,-2)
      CALL PLOT(0.,-8.30,-2)  $  CALL PLOT(-5.8,0.,-2)
      CALL SYMBOL(.5,-.2,.1,ID(13),0.,50) $ CALL PLOT(5.3,.75,-3)
      CALL AXIS(0.,0.,ID(9),-20,7.,90.,X(N+1),X(N+2))
      CALL AXIS(0.,0.,ID(11),20,5.,180.,Y(N+1),Y(N+2))
30    Y(N+2)=-Y(N+2)             $ CALL LINE(Y,X,N,1,NP,NS)
      Y(N+2)=-Y(N+2)             $ CALL PLOT(1.85,-2.10,-3)
      RETURN                     $ END
      SUBROUTINE AXIS(X0,Y0,L,NC,RL,ANG,RMIN,DR)
      DIMENSION L(1) $ A=ANG*3.14159/180. $ DX=.1*COS(A) $ DY=.1*SIN(A
      IC=ISIGN(1,NC) $ NNC=IABS(NC) $ R=.1 $ N=1 $ X=X0 $ Y=Y0$
10    CALL PLOT(X,Y,3) $ X=X+DX $ Y=Y+DY $ CALL PLOT(X,Y,2)
      CALL PLOT(X-.21*DY*IC,Y+.21*DX*IC,2)
      IF(N.EQ.5) CALL PLOT(X-.42*DY*IC,Y+.42*DX*IC,2)
      IF(N.EQ.10) CALL PLOT(X-.70*DY*IC,Y+.70*DX*IC,2)
      N=MOD(N,10)+1 $ R=R+.1 $ IF(R.LT.RL) GO TO 10
      A=ANG-(IC+1)*45. $ DX=10.*DX $ DY=10.*DY
      C=-.175+.125*IC      $      D=.19+.35*IC
      X=X0+C*DX-D*DY       $      Y=Y0+C*DY+D*DX
      R=AMAX1(ABS(RMIN),ABS(RMIN+DR*RL)) $ R=ALOG10(R)
      IR=INT(ABS(R)) $ IF(R.LT.0.) IR=-(IR+1) $ IR=IR-MOD(IR,3)
      R1=RMIN/10.**IR $ DR1=DR/10.**IR $ R=0.
20    ENCODE(7,101,S)R1 $ CALL SYMBOL(X,Y,.07,S,A,7) $ R1=R1+DR1
      X=X+DX $ Y=Y+DY $ R=R+1. $ IF(R.LE.RL) GO TO 20
      R=(RL-.1*NNC)/2.      $      C=.1+.5*IC
      X=X0+R*DX-C*DY        $      Y=Y0+R*DY+C*DX
      CALL SYMBOL(X,Y,.1,L,ANG,NNC) $ IF(IR.EQ.0) RETURN
      ENCODE(5,102,S) $ CALL SYMBOL(999.,999.,.10,S,ANG,5)
      CALL WHERE(X,Y,A)
      ENCODE(3,103,S) IR  $ CALL SYMBOL(X,Y,.07,S,ANG,3)
101   FORMAT(F7.2)
102   FORMAT(5H  *10)
103   FORMAT(I3)
      RETURN     $     END
C     =     SUBROUTINE SCALE(DATA,LENGTH,N,K)                        =
C     =                                                              =
C     =  REAL DATA = N+2 DIMENSIONED ARRAY OF DATA TO BE SCALED      =
C     =  INTEGER N = NUMBER OF DATA POINTS                           =
C     =  REAL LENGTH = LENGTH OF THE PLOT AXIS (E.G. IN INCHES)      =
C     =  INTEGER K = UNUSED PARAMETER INCLUDED FOR COMPATIBILITY     =
```

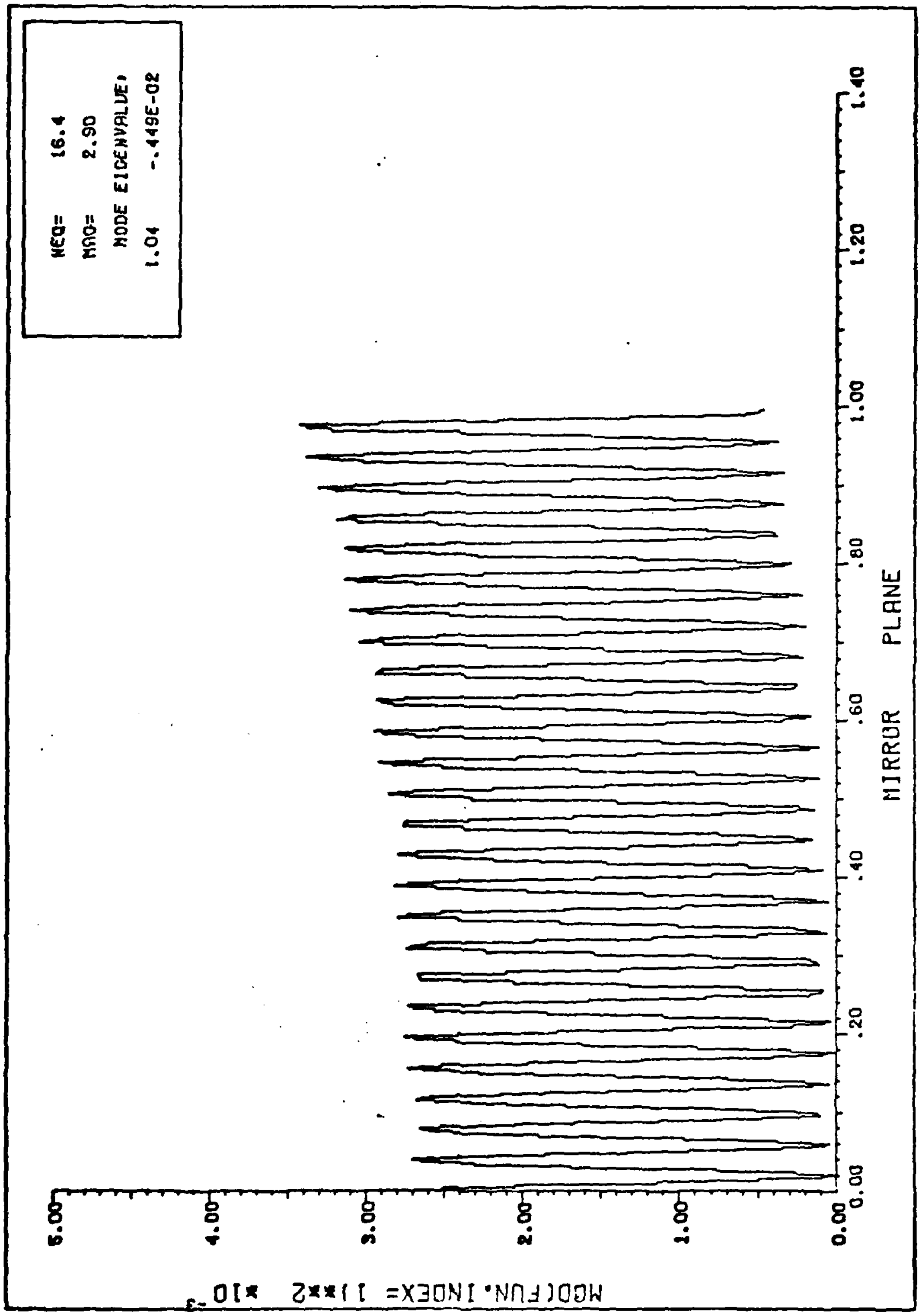
```
C     =              WITH THE EQUIVALENT CALCOMP SUBROUTINE         =
C     =                                                              =
C     =  THE FOLLOWING VALUES ARE RETURNED:                          =
C     =                                                              =
C     =         DATA(N+1) = ADJUSTED DATA MINIMUM                    =
C     =         DATA(N+2) = "NICE" SCALE FACTOR IN DATA UNITS        =
C     =                     PER LENGTH UNIT (E.G. VOLTS/INCH)        =
C     ================================================================

      SUBROUTINE SCALE(DATA,LENGTH,N,K)

      REAL DATA(N), LENGTH, SF(5)
      DATA SF /1., 2., 2.5, 5., 10. /

C     COMPUTE THE RAW SCALE FACTOR

      DMIN=DMAX=DATA(1)
      DO 10 I=1,N
            IF(DATA(I) .LT. DMIN) DMIN = DATA(I)
            IF (DATA(I). GT. DMAX) DMAX = DATA(I)
10    CONTINUE

C     --------------------------------
C     EXCLUDE TRIVIAL ERROR CASES
      DATA(N+1) = DMIN
      DATA(N+2) = 1.0
      IF (LENGTH .LE. 0.0 .OR. DMAX .EQ. DMIN )  RETURN
C     --------------------------------

      RAWSF = (DMAX - DMIN) / LENGTH

C     RAWSF = SFMANT * 10. ** SFEXP, WHERE 1 .LE. SFMANT .LT. 10

      SFEXP = AINT( ALOG10( RAWSF ) )
      IF ( RAWSF .LT. 1.0 ) SFEXP = SFEXP - 1.0
      SFMANT = RAWSF * 10.0 ** (-SFEXP)

C     LOCATE NEXT LARGER "NICE" SCALE FACTOR

      DO 20 I=1,5
20    IF ( SF(I) .GT. SFMANT )  GO TO 30
      PRINT*," SCALE: SCALE FACTOR ERROR ... "  $  RETURN

30    SFNICE = SF(I) * 10.0 ** SFEXP

C     COMPUTE ADJUSTED DATA MINIMUM

      ADJMIN = AINT ( DMIN / SFNICE ) * SFNICE
      IF ( ADJMIN .GT. DMIN ) ADJMIN = ADJMIN - SFNICE

      IF ( (DMAX - ADJMIN) / SFNICE .LE. LENGTH )  GO TO 40
```

```
C     NEED TO USE THE NEXT LARGER SCALE FACTOR

      IF ( I .LT. 5 )  SFNICE = SF(I+1) * 10.0 ** SFEXP
      IF ( I .EQ. 5 )  SFNICE = 20.0    * 10.0 ** SFEXP
      ADJMIN = AINT ( DMIN / SFNICE ) * SFNICE
      IF ( ADJMIN .GT. DMIN) ADJMIN = ADJMIN - SFNICE

40    CONTINUE

      DATA(N+1) = ADJMIN
      DATA(N+2) = SFNICE

      RETURN
      END
      COMPLEX FUNCTION CERF(ZZ)
      COMPLEX ZZ,Z,A,A1,A2,B,B1,B2,F,F1
      Z=ZZ
      IF(CABS(Z).GE.3.)GOTO30
      J=0.
      A=Z
      B=Z
   10 J=J+1
      B=-Z*Z*CMPLX(FLOAT(2*J-1),0.)*B
      B=B/CMPLX(FLOAT(J  ),0.)/CMPLX(FLOAT(2*J+1),0.)
      A=A+B
      IF(J.GE.1000)GOTO50
      IF(CABS(B/A).GE.(1.E-10)) GO TO 10
      CERF=(1.128379167,0.)*A
      RETURN
   30 IF(REAL(ZZ).LT.0.)Z=-ZZ
      A2=(1.,0.)
      B2=Z
      F2=A2/B2
      A1=Z
      B1=Z*Z+(0.5,0.)
      F1=A1/B1
      J=1
   40 J=J+1
      A=Z*A1+CMPLX(FLOAT(J)/2.,0.)*A2
      B=Z*B1+CMPLX(FLOAT(J)/2.,0.)*B2
      F=A/B
      IF(J.GT.1000)GOTO 50
      IF(CABS((F-F1)/F).LT.(1.E-10))GOTO60
      A2=A1
      B2=B1
      A1=A
      B1=B
      F1=F
      GOTO40
50    WRITE(8,99)
```

```
99 FORMAT(    "  ERROR FUNCTION ROUTINE DID NOT CONVERGE ")
   IER=1
   RETURN
60 F1=(0.5,0.)*CEXP(-Z*Z)*F
   CERF=1.128379167*F1
   CERF=1.-CERF
   IF(REAL(ZZ).LT.0.) CERF=-CERF
70 RETURN
   END
   COMPLEX FUNCTION FRESL(X)
   COMPLEX EYE,Z,CERF
   EYE=(0.,1.) $ PI=2.*ASIN(1.)
   Z=SQRT(PI)*X*(1.-EYE)/2.
   FRESL=(1.+EYE)/2.*CERF(Z)
   FRESL=CONJG(FRESL)
   RETURN $ END
```

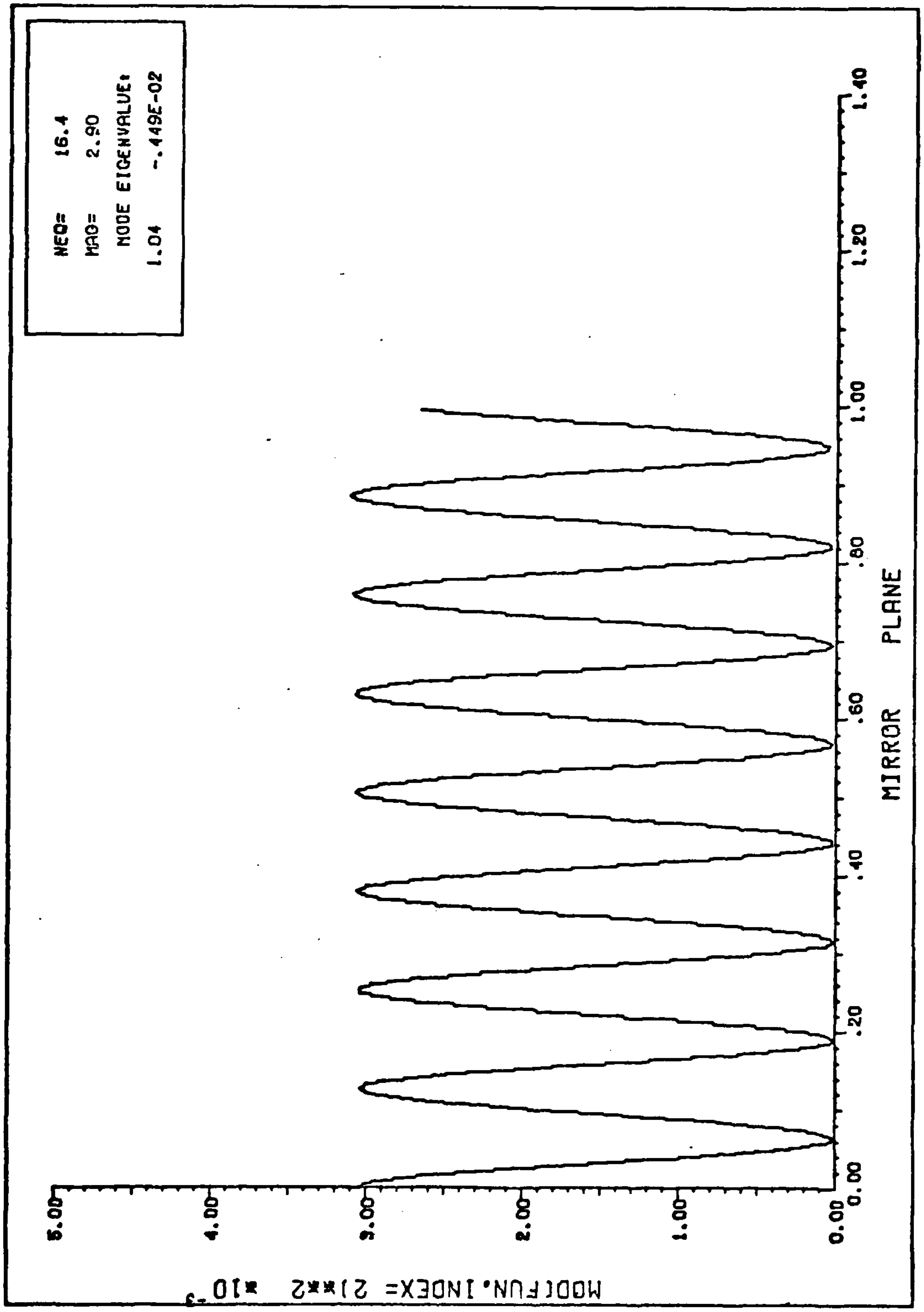
Appendix E

This appendix displays plots of the intensity of the function $M_n(x)$ for $n=1$ through 8, for bare resonator parameters of $M=2.9$, $N_f=16.4$.



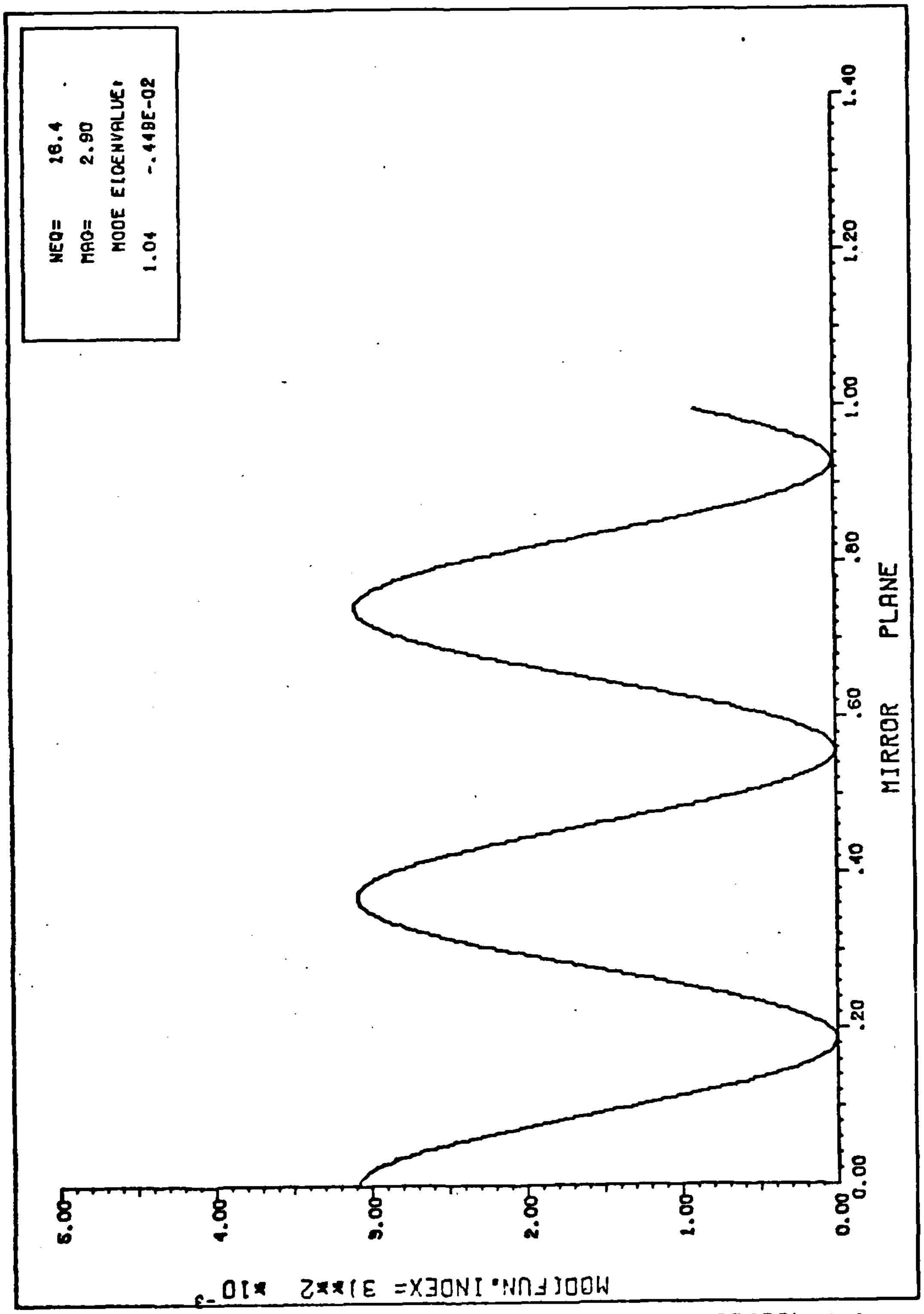
EVEN PARITY MODE # 1 APPROX # 1

Figure 30



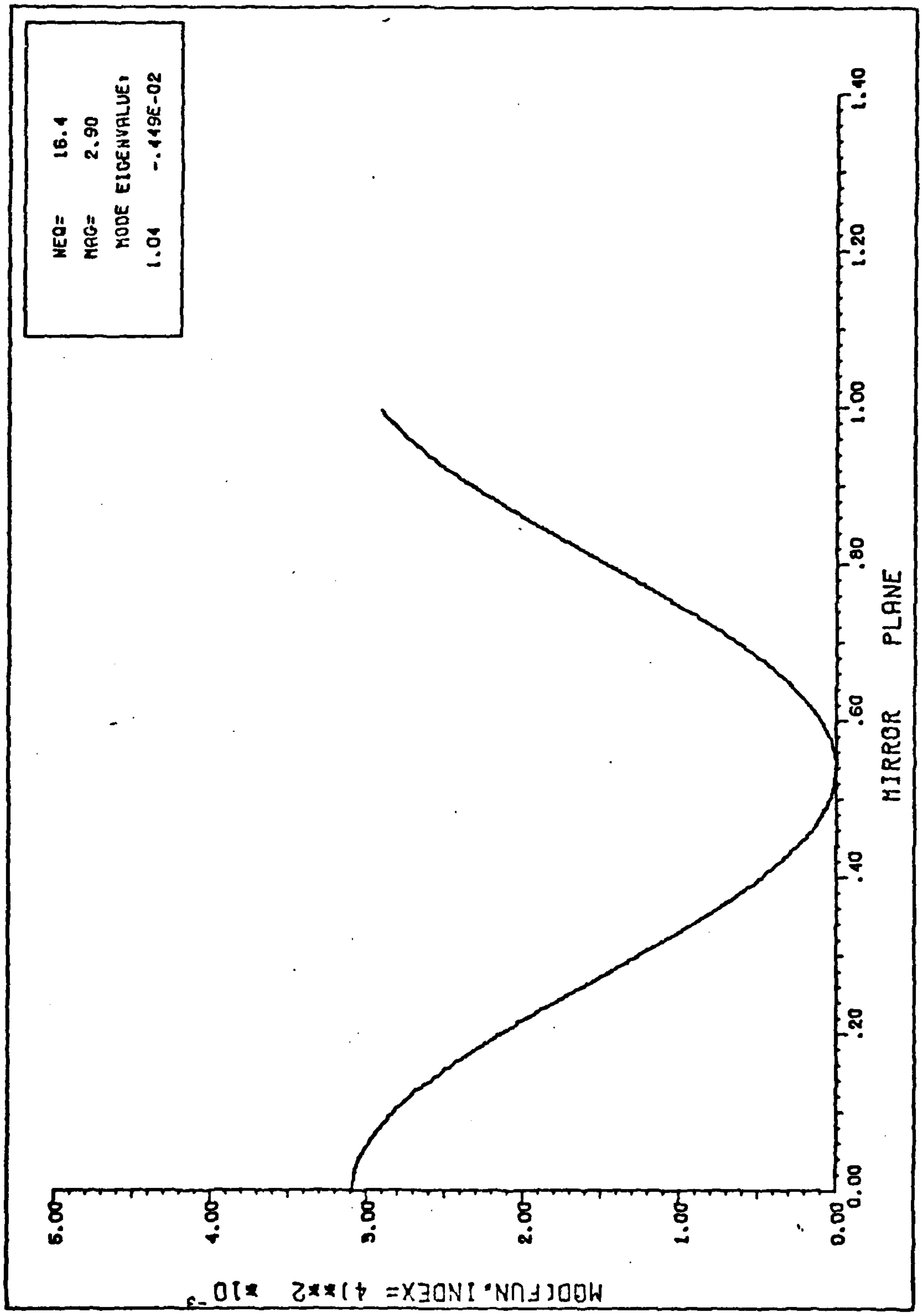
EVEN PARITY MODE # 1 APPROX # 1

Figure 31



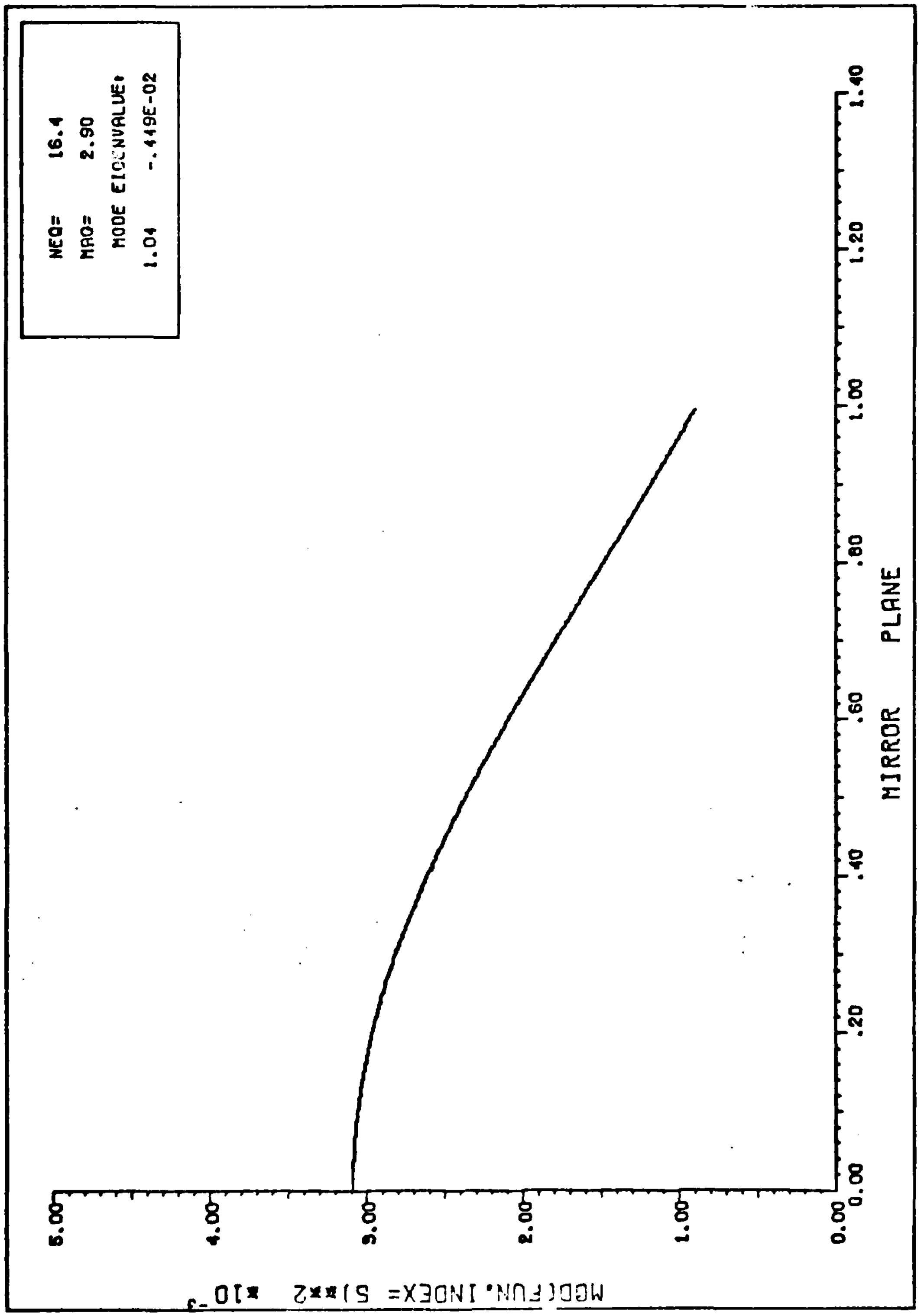
EVEN PARITY MODE # 1 APPROX # 1

Figure 32



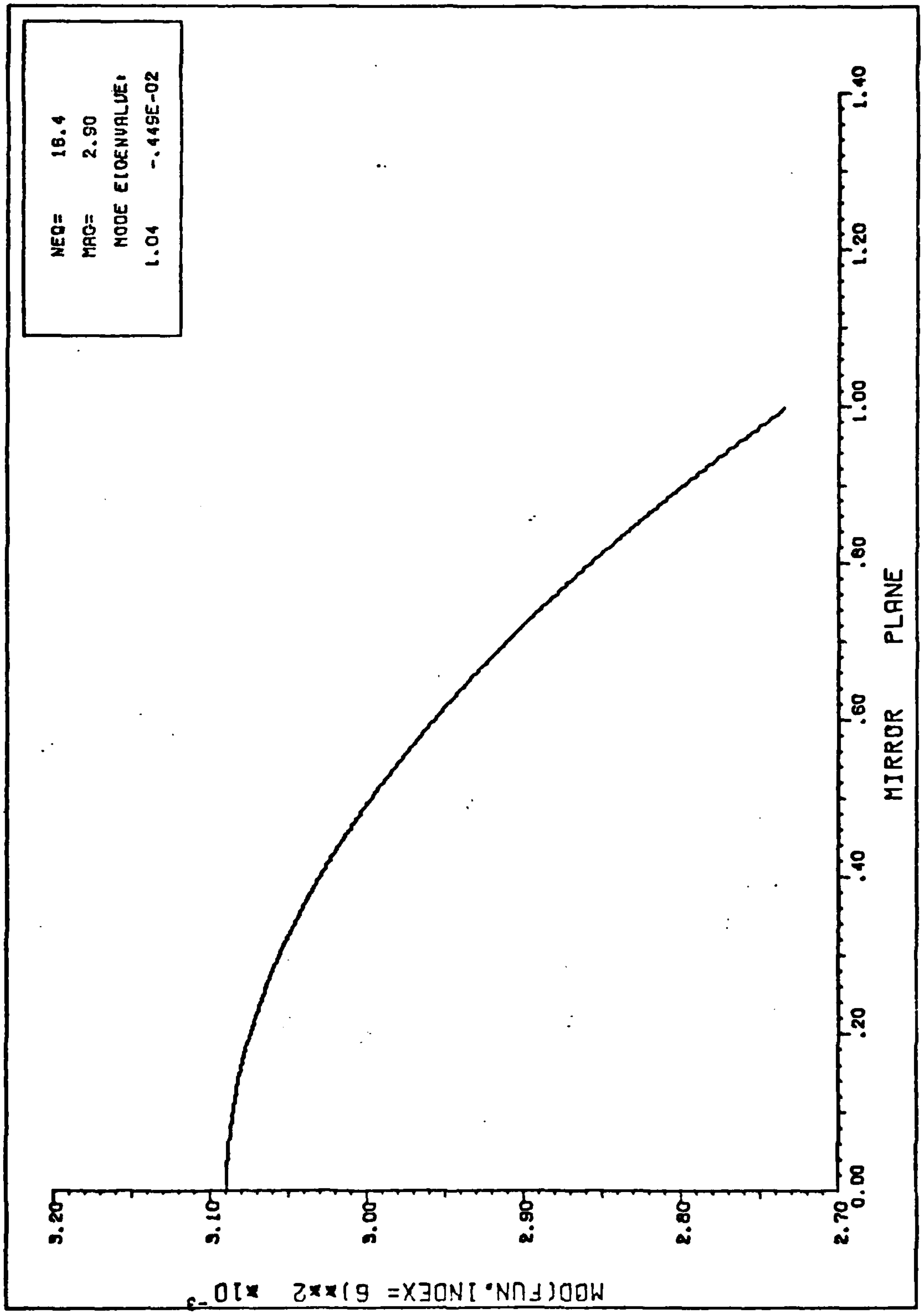
EVEN PARITY MODE # 1 APPROX # 1

Figure 33



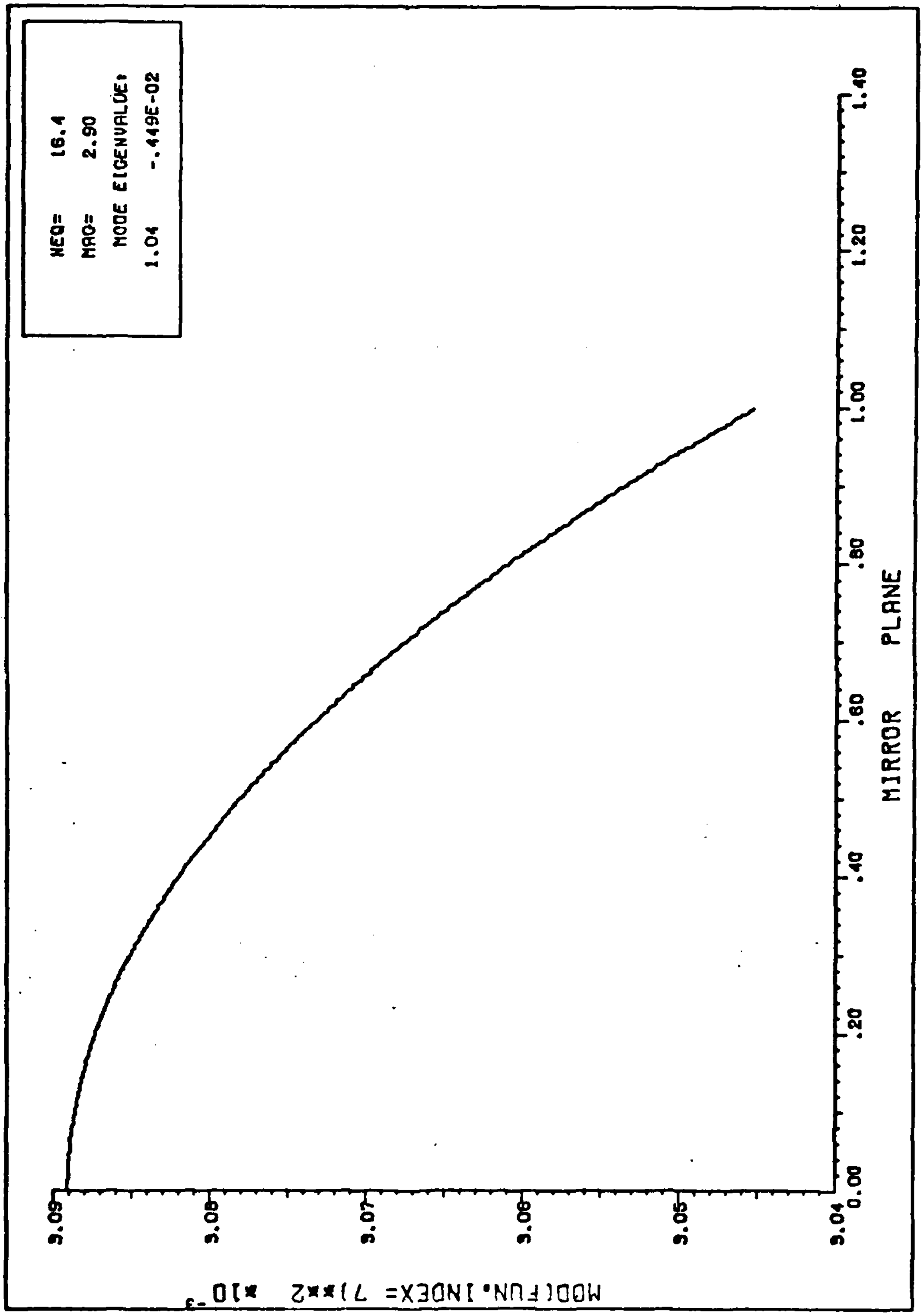
EVEN PARITY MODE # 1 APPROX # 1

Figure 34



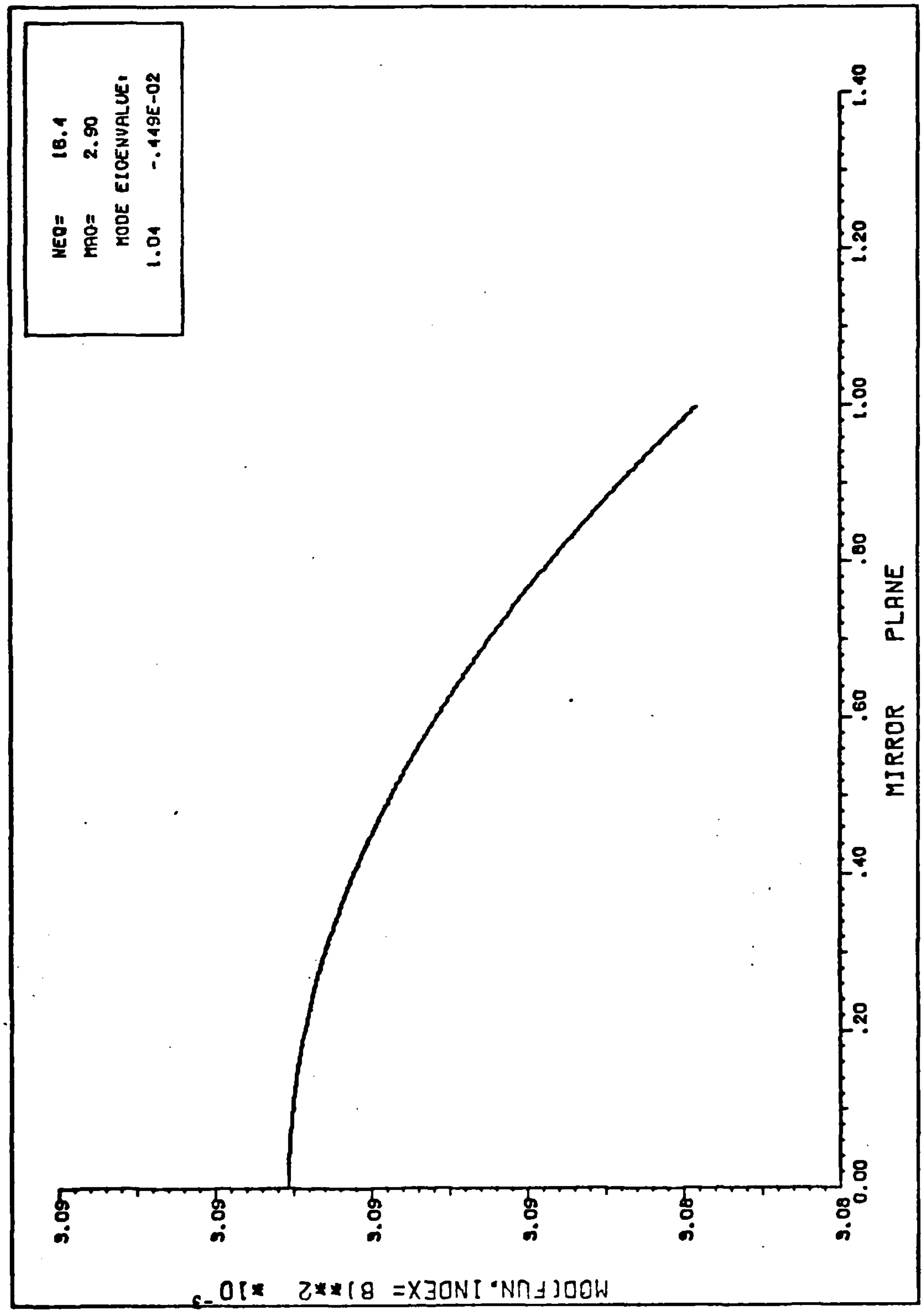
EVEN PARITY MODE # 1 APPROX # 1

Figure 35



EVEN PARITY MODE # 1 APPROX # 1

Figure 36



EVEN PARITY MODE # 1 APPROX # 1

Figure 37

VITA

James E. Rowley was born in Waterbury, Connecticut on 17 May 1957. He graduated from Wamogo Regional High School in Litchfield, Connecticut in June 1975. He then attended Norwich University in Northfield, Vermont from 27 August 1975 until 27 May 1979 when he graduated with a B.S. in Physics and was commissioned in the Air Force. His first active duty assignment was to the Air Force Institute of Technology at Wright-Patterson AFB.

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The mode eigenvalue equation for an unstable strip laser resonator is developed from scalar diffraction theory. The field distributions are expressed as a series and the integral is then evaluated using a first order approximation to the method of stationary phase. The resulting approximate closed form is rearranged to form an eigenvalue polynomial, the roots of which are the mode eigenvalues. Eigenfunction expressions are then developed using sec-

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ond order approximation to the method of stationary phase. Modifications to these expressions are then made to account for the presence of uniform gain in the resonator.

The results of a computer program using the derived expressions are presented. Comparisons to previously published results are made for the bare cavity case, and results for the loaded cavity case follow.

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